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Certifying solutions to square systems of polynomial-exponential equations

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ABSTRACT

Smale's α-theory certifies that Newton iterations will converge quadratically to a solution of a square system of analytic functions based on the Newton residual and all higher order derivatives at the given point. Shub and Smale presented a bound for the higher order derivatives of a system of polynomial equations based in part on the degrees of the equations. For a given system of polynomial-exponential equations, we consider a related system of polynomial-exponential equations and provide a bound on the higher order derivatives of this related system. This bound yields a complete algorithm for certifying solutions to polynomial-exponential systems, which is implemented in `alphaCertified`. Examples are presented to demonstrate this certification algorithm.

1. Introduction

A map $f : \mathbb{C}^n \to \mathbb{C}^n$ is called a square system of polynomial-exponential functions if f is polynomial in both the variables $x_1, \ldots, x_n$ and finitely many exponentials of the form $e^{\beta x_i}$ where $\beta \in \mathbb{C}$. That is, there exists a polynomial system $P : \mathbb{C}^{n+m} \to \mathbb{C}^n$, analytic functions $g_1, \ldots, g_m : \mathbb{C} \to \mathbb{C}$, and integers $\sigma_1, \ldots, \sigma_m \in \{1, \ldots, n\}$ such that

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$$f(x_1,\ldots,x_n) = P(x_1,\ldots,x_n, g_1(x_{\sigma_1}),\ldots,g_m(x_{\sigma_m}))$$

where each g_i satisfies some linear homogeneous partial differential equation (PDE) with complex coefficients. In particular, for each i, there exists a positive integer r_i and a linear function $\ell_i : \mathbb{C}^{r_i+1} \to \mathbb{C}$ such that $\ell_i(g_i, g_i', \ldots, g_i^{(r_i)}) = 0$.

Consider the polynomial-exponential system $\mathcal{F} : \mathbb{C}^{n+m} \to \mathbb{C}^{n+m}$ where

$$\mathcal{F}(x_1,\ldots,x_n,y_1,\ldots,y_m) = \begin{bmatrix} P(x_1,\ldots,x_n,y_1,\ldots,y_m) \\ y_1 - g_1(x_{\sigma_1}) \\ \vdots \\ y_m - g_m(x_{\sigma_m}) \end{bmatrix}. \tag{1}$$

Since the projection map $(x, y) \mapsto x$ defines a bijection between the solutions of $\mathcal{F}(x, y) = 0$ and $f(x) = 0$, we will only consider certifying solutions to square systems of polynomial-exponential equations of the form $\mathcal{F}(x, y) = 0$.

For a square system $g : \mathbb{C}^n \to \mathbb{C}^n$ of analytic functions, a point $x \in \mathbb{C}^n$ is an *approximate solution* of $g = 0$ if Newton iterations applied to x with respect to g quadratically converge immediately to a solution of $g = 0$. The certificate returned by our approach that a point is an approximation solution of $\mathcal{F} = 0$ is an α-theoretic certificate. In short, α-theory, which started for systems of analytic equations in Smale (1986), provides a rigorous mathematical foundation for the fact that if the Newton residual at the point is small and the higher order derivatives at the point are controlled, then the point is an approximate solution. For polynomial systems, by exploiting the fact that there are only finitely many nonzero derivatives, Shub and Smale (1993) provide a bound on all of the higher order derivatives. For polynomial-exponential systems, our approach uses the structure of $\mathcal{F}$ together with the linear functions ℓ_i to bound the higher order derivatives.

Systems of polynomial-exponential functions naturally arise in many applications including engineering, mathematical physics, and control theory, to name a few. On the other hand, such functions are typical solutions to systems of linear partial differential equations with constant coefficients. Systems, including ubiquitous functions like $\sin(x)$, $\cos(x)$, $\sinh(x)$, and $\cosh(x)$, can be equivalently reformulated as systems of polynomial-exponential functions, since these functions can be expressed as polynomials involving $e^{\beta x}$ for suitable $\beta \in \mathbb{C}$. Since computing all solutions to such systems is often nontrivial, methods for approximating and certifying some solutions for general systems is very important, especially in the aforementioned applications.

In the rest of this section, we introduce the needed concepts from α-theory. Section 2 formulates the bounds for the higher order derivatives of polynomial-exponential systems and presents a certification algorithm for polynomial-exponential systems. In Section 3, we discuss methods for generating numerical approximations to solutions of polynomial-exponential systems. Section 4 describes the implementation of the certification algorithm in `alphaCertified` as well as demonstrating the algorithms on a collection of examples. Appendix A demonstrates the input, command-line execution, and output of `alphaCertified` for a polynomial-exponential system from Section 4.1. Files for all of the examples are available at www.nd.edu/~jhauenst/PolyExp.

1.1. Smale's α-theory

We provide a summary of the elements of α-theory used in the remainder of the article as well as in `alphaCertified`. Hence, this section closely follows (Hauenstein and Sottile, 2012, § 1) with "polynomial" replaced by "analytic." We focus on *square* systems, which are systems with the same number of variables and functions, with more details provided in Blum et al. (1998).

Let $f : \mathbb{C}^n \to \mathbb{C}^n$ be a system of analytic functions. The set of zeros of f is $\mathcal{V}(f) = \{\xi \in \mathbb{C}^n \mid f(\xi) = 0\}$ with $Df(x)$ being the Jacobian matrix of f at x. For a point $x \in \mathbb{C}^n$, the point $N_f(x)$ is called the *Newton iteration of f at x* where the map $N_f : \mathbb{C}^n \to \mathbb{C}^n$ is defined by

$$N_f(x) = \begin{cases} x - Df(x)^{-1} f(x) & \text{if } Df(x) \text{ is invertible,} \\ x & \text{otherwise.} \end{cases}$$