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Eliminating dual spaces

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ABSTRACT

Macaulay dual spaces provide a local description of an affine scheme and give rise to computational machinery that is compatible with the methods of numerical algebraic geometry. We introduce eliminating dual spaces, use them for computing dual spaces of quotient ideals, and develop an algorithm for detection of embedded points on an algebraic curve.

1. Introduction

We explore algorithms that use dual spaces to compute local information about polynomial ideals over the complex numbers, as an alternative to working directly with polynomials. This strategy often has advantages in the regime of approximate numerical computations and for hybrid symbolic-numerical algorithms. Many computations are reduced to linear algebra, allowing numerical linear algebra techniques to be applied.

Given generators of a polynomial ideal I and a point p in its vanishing set, the dual space of I at p is the vector space dual of the extension of I in the local ring at p, and it uniquely encodes the local properties of I there. Certain combinatorial information about the dual space, such as dimension, can be accurately computed even when p is only known approximately but with high enough precision. Existing methods of numerical algebraic geometry (see e.g., Sommese et al., 2005 and Sommese and Wampler II, 2005) offer efficient algorithms to approximately compute points on the vanishing set of an ideal.

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The contributions of this paper include:

- Establishing a correspondence between a local monomial order on a local ring (primal order) and the corresponding dual order on the monomials of the dual space.
- Introducing the notion of *eliminating dual spaces* as a computational tool.
- Applying these ideas to construct an algorithm for detecting embedded primary components of a curve.

Additionally, we consolidate the necessary background on dual spaces, state their properties in a modern language, and provide proofs where references are unavailable.

The idea of studying systems of polynomials through dual spaces dates back to Macaulay (1916). Most of the recent work using Macaulay's machinery concerns zero-dimensional ideals or, geometrically speaking, isolated points. This includes algorithms for computing a basis of the dual space (Mourrain, 1997; Dayton and Zeng, 2005) and the local Hilbert function at an isolated point (Griffin et al., 2011), as well as various deflation procedures (Lecerf, 2002; Leykin et al., 2006; Hauenstein and Wampler, 2013). Several studies depart from the zero-dimensional setting: the local dimension test (Bates et al., 2009), computations using dual spaces for homogeneous ideals (Hauenstein, 2011), and a numerical algorithm for local Hilbert polynomial in the general case (Krone and Leykin, 2014b).

As an application of eliminating dual spaces we present an algorithm for detecting embedded components of a curve. This fills in a key missing piece in the program to numerically compute a primary decomposition of a polynomial ideal as laid out in Leykin (2008) in the case of one-dimensional ideals. We remark that eliminating dual spaces may, in theory, be used to provide an embedded component test in dimension more than one. However, we did not pursue this direction since for practical computation this potential technique appears to be inferior to an alternative treatment we develop in Krone and Leykin (2014b). Our algorithms for dual spaces and detecting embedded components are implemented in the *Macaulay2* computer algebra system (Grayson and Stillman, 2002) and the code is posted at Krone and Leykin (2014a).

The rest of the paper is structured as follows. Section 2 gives an expository background on dual spaces of ideals in a local ring, laying out the facts that will be used later in the paper and giving proofs where they were not readily available in the literature. Additionally we characterize the dual space of quotient ideals. Section 3 describes the correspondence between the initial terms of an ideal with respect to a (primal) local monomial order and the initial terms of the dual space in the dual order. Such a relation has been observed before in special settings such as zero-dimensional ideals with a graded order, but we give a fully general result. This section also describes the relation between truncated dual spaces and the Hilbert function. Section 4 introduces the eliminating dual space, which generalize the truncated dual space, and can be used to compute truncated dual spaces of a quotient ideal. These ideas are applied in Section 5 to the problem of detecting the embedded components on a curve, and examples are provided. Section 6 gives some concluding remarks.

2. Macaulay dual spaces

For $\alpha \in (\mathbb{Z}_{\geq 0})^{\mathrm{N}}$ and $y \in \mathbb{C}^N$, define

- $x^\alpha = x_1^{\alpha_1} \cdots x_N^{\alpha_N}$,
- $|\alpha| = \sum_{i=1}^N \alpha_i$,
- $\alpha! = \alpha_1!\alpha_2! \ldots \alpha_N!$,
- $\partial^\alpha = \frac{1}{\alpha!}\frac{\partial^{|\alpha|}}{\partial x^\alpha}$, and
- $\partial^\alpha[y] : R \to \mathbb{C}$ defined by $\partial^\alpha[y](g) = (\partial^\alpha g)(y)$.

The differential functional $\partial^\alpha[y]$ sometimes would be written $\partial^{x^\alpha}[y]$ and when the point y is implied $\partial^\alpha[y]$ would be written as ∂^α; e.g., $\partial^1 - \partial^y + \partial^{x^2yz}$. For $y \in \mathbb{C}^N$, let $D_y = \mathrm{span}_{\mathbb{C}}\left\{\partial^\alpha[y] \mid \alpha \in (\mathbb{Z}_{\geq 0})^{\mathrm{N}}\right\}$

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