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Bounding the number of limit cycles for a polynomial Liénard system by using regular chains ☆

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A R T I C L E I N F O

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A B S T R A C T

In this paper, we study the bound of the number of limit cycles by Poincaré bifurcation for a Liénard system of type $(4, 3)$. An automatic algorithm is constructed based on the Chebyshev criteria and the tools of regular chain theory in polynomial algebra. We prove the system can bifurcate at most 6 limit cycles from the periodic annulus by this algorithm and at least 4 limit cycles by asymptotic expansions of the related Melnikov functions.

1. Introduction

The second part of Hilbert's 16th problem (Hilbert, 1902) asks for the maximal number of limit cycles and its distribution of the following polynomial system of degree n:

$$\dot{x} = P_n(x, y), \dot{y} = Q_n(x, y). \quad (1)$$

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In order to reduce its difficulty, mathematicians study this problem in two directions, concerning small limit cycles near singular points (local Hilbert's 16th problems, see Zoladek, 1995) and concerning simpler forms of (1):

Smale (1998) restricted Hilbert's 16th problem to the following Liénard system:

$$\dot{x} = y - f(x), \ \ \dot{y} = -x. \tag{2}$$

Arnold (1977) proposed to study the maximal number of zeros of the following Melnikov function (weak Hilbert's 16th problem):

$$M(h) = \oint_{\Gamma_h} q(x, y)dx - p(x, y)dy, \ \ h \in J, \tag{3}$$

which corresponds to a simpler form of (1):

$$\dot{x} = H_y + \varepsilon p(x, y), \ \ \dot{y} = -H_x + \varepsilon q(x, y), \tag{4}$$

where $H(x, y)$, $p(x, y)$ and $q(x, y)$ are polynomials with $\max\{deg(p), deg(q)\} = deg(H) - 1 = n$, $H(x, y) = h$ (h inside an open interval J) defines at least one family of closed curves Γ_h, and ε is a very small and positive parameter. Poincaré–Pontryagin Theorem (see lemma 3.2 in Pontryagin, 1934) told us that $M(h)$ is the first-order approximation of Poincaré map of (4), and the number of zeros of $M(h)$ provides an upper bound of the number of limit cycles. However, weak Hilbert's 16th problem is as difficult as its original form. There is no general method for a general system (4) and almost all of its results are on concrete and special forms of (4).

One interesting form is obtained by reducing (4) and generalising (2),

$$\dot{x} = y, \ \ \dot{y} = g(x) + \varepsilon f(x)y \tag{5}$$

which is called a Liénard system of type (m, n), here $g(x)$ and $f(x)$ are polynomials of degree respectively m and n.

On one hand, system (5) can be applied in many natural fields such as Newtonian mechanical problems, chemical reactions, vacuum tube technology and so on. On the other hand, studying the number of limit cycles or zeros of Melnikov function of (5) is the first step to find the limit cycles of general system (1) and (4). For instance, the author in Li et al. (2009) proved a cubic planar polynomial system has 13 limit cycles, which is one best result on Hilbert's 16th problem for cubic systems. This cubic system was obtained by "tremble-making" from a cubic system (5) of type $(3, 2)$. Therefore, to detect the number of limit cycles of (2) for some exact m and n is a hot research topic.

For type $(3, 2)$, Dumortier and Li (2001a, 2001b, 2003a, 2003b) have presented a complete study on 5 cases according to the portraits of unperturbed systems. They obtained different sharp bounds of the number of their limit cycles by studying zeros of Melnikov functions. Later, Li et al. proved 2 is an upper bound for a symmetric system of type $(3, 2)$. For type $(5, 4)$, many Liénard systems with symmetry are investigated, see Zhang et al. (2006), Asheghi and Zangeneh (2008a, 2008b, 2010), Kazemi et al. (2012), Sun (2012), Sun et al. (2014), Qi and Zhao (2012), Zhao (2014) and their references. For both types, the related Melnikov functions consist of at most three generating elements, while there exist 4 generating elements for the corresponding Melnikov functions for type $(4, 3)$. The classical methods of Picard–Focus equation theory used before cannot be applied to this case. There existed no results for type $(4, 3)$ until Grau et al. (2011) proposed the Chebyshev criteria. This algebraic criteria can deal with Melnikov functions with more than 3 generating elements. Based on this criteria, Wang and Xiao (2011) and Wang (2012) studied the following system of type $(4, 3)$

$$\dot{x} = y, \ \ \dot{y} = x(x-1)^3 + \varepsilon(a + bx + cx^2 + x^3)y, \tag{6}$$

and proved that the system has at most 4 limit cycles. Atabaigi and Zangeneh (2011) proved system

$$\dot{x} = y, \ \ \dot{y} = x(x+1)^2(x - \frac{2}{3}) + \varepsilon(a + bx + cx^2 + x^3)y, \tag{7}$$

has at most 3 limit cycles, and 3 limit cycles can be reached.

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