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A Henneberg-based algorithm for generating tree-decomposable minimally rigid graphs

Marta R. Hidalgo [a], Robert Joan-Arinyo [b]

[a] *Systems Genomics Lab., Centro de Investigación Príncipe Felipe, València, Spain*
[b] *GIE, Universitat Politècnica de Catalunya, Barcelona, Catalonia*

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ABSTRACT

In this work we describe an algorithm to generate tree-decomposable minimally rigid graphs on a given set of vertices V. The main idea is based on the well-known fact that all minimally rigid graphs, also known as Laman graphs, can be generated via Henneberg sequences. Given that not each minimally rigid graph is tree-decomposable, we identify a set of conditions on the way Henneberg steps are applied so that the resulting graph is tree-decomposable. We show that the worst case running time of the algorithm is $O(|V|^3)$.

1. Introduction

We address the problem of generating tree-decomposable minimally rigid graphs, also known as Laman graphs, by applying sequences of Henneberg constructions on a given set of vertices.

This kind of graphs is of interest in graph-based geometric constraint solving and its applications in many different fields, such as computer-aided design, molecular modelling, tolerance analysis and theorem proving. In graph-based geometric constraint solving technology, the problem is defined as a rough sketch of an object made out of simple geometric elements. Then the user selects the intended exact shape by annotating the sketch with constraints. The resulting annotated sketch is captured as a graph where vertices are geometric elements and edges are the constraints. Finally, a geometric constraint solver checks whether the set of geometric constraints coherently defines the object and, if so, determines the position of the geometric elements.

E-mail addresses: mhidalgo@cipf.es (M.R. Hidalgo), robert@cs.upc.edu (R. Joan-Arinyo).

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The success of the geometric constraint solver depends to a great extent on the combinatorial properties of the graph. If the graph is minimally rigid, the geometric constraint problem defines a rigid object and, consequently, the solution to the constraint problem has finitely many solution instances (Hoffmann and Joan-Arinyo, 2005). If the graph is tree-decomposable, tools developed in graph-based geometric constraint solving can be applied to solve the constraint problem at hand (Hoffmann et al., 2001a, 2001b; Jerman et al., 2006; Joan-Arinyo and Soto, 1997; Joan-Arinyo et al., 2001).

It is well known that the set of graphs generated by Henneberg sequences and the set of minimally rigid graphs are the same sets (Crapo and Whiteley, 1993; Graver et al., 1993; Tay and Whiteley, 1985; Whiteley, 1996). However, not every minimally rigid graph is tree-decomposable (Fudos and Hoffmann, 1997; Owen and Power, 2006). Hence, the idea that guided this work was to find out conditions on the application of Henneberg construction steps so that the resulting graph is minimally rigid and tree-decomposable.

In this paper we present a theory that characterizes tree-decomposable minimally rigid graphs by an inductive construction of Henneberg steps. Then we describe an algorithm for generating tree-decomposable minimally rigid graphs of a given order based on this theory.

The rest of the paper is organized as follows. In Section 2 we recall known theoretical results that we will use later on. In Section 3 we provide new theoretical results that characterize a class of Henneberg sequences which generates tree-decomposable minimally rigid graphs. Section 4 is devoted to describing the algorithm that actually builds this kind of graphs on a set of given vertices V. The algorithm implements the results of the previous section. We show that the algorithm's worst case running time is $O(|V|^3)$. We provide some conclusions in Section 5. Finally, proofs for the theorems in the manuscript are developed in Section 6.

2. Preliminaries

In this section we describe tools that will be used later on. First we define the concept of graph tree-decomposability and recall Henneberg constructions. Then we formalize sequences of Henneberg constructions as rewrite systems. Finally we recall a characterization of minimally rigid graphs.

2.1. Tree-decomposable graphs

Consider the graph $G = (V, E)$ and let $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ and $G_3 = (V_3, E_3)$ be three subgraphs of G such that

$$V = V_1 \cup V_2 \cup V_3, \qquad E = E_1 \cup E_2 \cup E_3$$

and the set of vertices pairwise share one vertex

$$V_1 \cap V_2 = \{a\}, \quad V_2 \cap V_3 = \{b\}, \quad V_3 \cap V_1 = \{c\}$$

We say that $\{G_1, G_2, G_3\}$ is a ternary decomposition of G induced by the vertices $\{a, b, c\}$. Fig. 1a shows a graph and Fig. 1b shows a ternary decomposition induced by the vertices $\{a, h, f\}$. In what follows we will refer to this set of vertices as *triple of hinges* or just as *triple*.

Definition 2.1. Let $G = (V, E)$ be a graph. A ternary tree T is a *tree-decomposition* of G if

1. G is the root of T,
2. Each node $G' \subseteq G$ of T is the father of exactly three nodes, say $\{G'_1, G'_2, G'_3\}$, which is a ternary decomposition of G', and
3. Each leaf node is the graph $(\{a, b\}, \{(a, b)\})$, that is, an edge (a, b) of $E(G)$.

A graph for which there is a tree-decomposition is called *tree-decomposable*.

In general, a tree-decomposition of a graph is not unique. Fig. 2 shows two different tree-decompositions for the graph given in Fig. 1a. For the sake of clarity, tree-decompositions only show the set of vertices within each node. The label on each tree edge is the triple of hinges that induces the ternary decomposition.

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