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On the maximum computing time of the bisection method for real root isolation

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ABSTRACT

The bisection method for polynomial real root isolation was introduced by Collins and Akritas in 1976. In 1981 Mignotte introduced the polynomials $A_{a,n}(x) = x^n - 2(ax-1)^2$, a an integer, $a \geq 2$ and $n \geq 3$. First we prove that if a is odd then the computing time of the bisection method when applied to $A_{a,n}$ dominates $n^5(\log d)^2$ where d is the maximum norm of $A_{a,n}$. Then we prove that if A is any polynomial of degree n with maximum norm d then the computing time of the bisection method, with a minor improvement regarding homothetic transformations, is dominated by $n^5(\log d)^2$. It follows that the maximum computing time of the bisection method is codominant with $n^5(\log d)^2$.

1. Introduction

We adopt the terminology "bisection method" for the polynomial real root isolation method introduced in Collins and Akritas (1976). In this paper we prove that when the method (using classical arithmetic for polynomial translations) is applied to the Mignotte polynomials (Mignotte, 1981), $A_{a,n}(x) = x^n - 2(ax-1)^2$ with a an odd integer, $a \geq 3$ and $n \geq 3$, its computing time dominates $n^5(\log d)^2$, where d is the max norm, $4a^2$, of $A_{a,n}$. (A quite similar result concerning the continued fractions method was recently proved in Collins (2016).)

In Section 2 we prove important theorems about the real and complex roots of $A_{a,n}$. In Section 3 we introduce notation to be used for polynomial transformations. In Section 4 we define a binary tree associated with the transformations performed on $A_{a,n}$, estimate its height, and show that it is isomorphic to a finite portion of the tree for computing the decimal expansion of $1/a$. In Section 5 we analyze the transformed polynomials, obtaining formulas for their coefficients. In Sections 6 and

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7 we analyze translation times at right and left nodes of the tree. In Section 8 we obtain the lower bound of $n^5(\log d)^2$ for the maximum computing time of the bisection method.

In Section 9 we discuss previous work on upper dominance bounds for the maximum computing time of the bisection method in Johnson (1998), Krandick (1995) and Eigenwillig et al. (2006). In Section 10 we introduce an improvement of the bisection method and prove that its maximum computing is dominated by $n^5(\log d)^2$.

2. The roots of $A_{a,n}$

We begin with some theorems about the real and complex roots of $A_{a,n}$.

Theorem 1. *$A_{a,n}$ has exactly three positive real roots, namely one in each of the intervals $(0, 1/a)$, $(1/a, 1)$ and $(1, \infty)$.*

Proof. $A_{a.n}(0) = -2 < 0$, $A_{a,n}(1/a) = 1/a^n > 0$, $A_{a,n}(1) = 1 - 2(a-1)^2 \leq 1 - 2 < 0$ and $A_{a,n}(\infty) > 0$ so there is at least one root in each of the named intervals. But $A_{a,n}$ has only three coefficient sign variations so by the Descartes rule each interval contains only one root. □

Let these three roots be called r_1, r_2 and r_3, $r_1 < r_2 < r_3$. Mignotte (1981) gave a sketchy and slightly incorrect proof that $r_1, r_2 \in (1/a - h, 1/a + h)$ where $h = a^{-n/2-1}$. In fact this is not true for $a = 2$ and $n \leq 7$. Krandick (1995) gave a correct proof of the following theorem,

Theorem 2. *Let $h = a^{-n/2-1}$. Then if $a \geq 3$ or if $a = 2$ and $n \geq 7$ then $r_1 \in (1/a - h, 1/a)$ and $r_2 \in (1/a, 1/a + h)$.*

In the following we will not use the case $a = 2$.

Theorem 3. *Let C be the circle of radius 1 centered at the origin. If $a \geq 2$ then r_1 and r_2 are the only roots of $A_{a,n}$ inside of C.*

Proof. We use Rouche's theorem, which can be found, for example, in Marden (1949). It states that if C is a simple closed Jordan curve and P(z) and $Q(x)$ are analytic inside of C and continuous on C and if $|P(z)| < |Q(z)|$ on C then $P(z) + Q(z)$ has the same number of zeros inside of C as $Q(z)$. We apply it with C the circle of radius 1 centered at the origin, $P(z) = z^n$ and $Q(z) = -2(az-1)^2$. Clearly $|P(z)| = 1$ everywhere on C and $|Q(z)| = 2|az-1|^2$ so it suffices to prove that $|az-1|^2 \geq 1$ on C. At $z = x + yi$,

$$|az-1|^2 = |(ax-1) + (ay)i|^2 = (ax-1)^2 + (ay)^2 = a^2x^2 - 2ax + 1 + a^2y^2.$$

On C, $y^2 = 1 - x^2$. Substituting we obtain

$$a^2x^2 - 2ax + 1 + a^2y^2 = -2ax + 1 + a^2.$$

Then, since $-1 \leq x \leq 1$ on C and $a \geq 2$,

$$-2ax + 1 + a^2 \geq -2a + 1 + a^2 = (a-1)^2 \geq 1. \qquad \square$$

Theorem 4. *$A_{a,n}$ is irreducible and therefore has no rational roots.*

Proof. The irreducibility of $A_{a,n}$ is a consequence of Eisenstein's irreducibility criterion using the prime number 2. □

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