ARTICLE IN PRESS YJSCO:1677

Journal of Symbolic Computation ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Symbolic Computation

www.elsevier.com/locate/jsc

Algorithmic calculus for Lie determining systems

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ARTICLE INFO

Article history:
Received 11 March 2014
Accepted 27 January 2016
Available online xxxx

MSC:
35-04
35N10
22-04
53-04

Keywords:
Determining equations
Lie symmetry
Lie algebra
Structure constants
Differential elimination
Algorithm

ABSTRACT

The infinitesimal symmetries of differential equations (DEs) or other geometric objects provide key insight into their analytical structure, including construction of solutions and of mappings between DEs. This article is a contribution to the algorithmic treatment of symmetries of DEs and their applications. Infinitesimal symmetries obey a determining system L of linear homogeneous partial differential equations, with the property that its solution vector fields form a Lie algebra $\mathcal{L}$. We exhibit several algorithms that work directly with the determining system without solving it. A procedure is given that can decide if a system specifies a Lie algebra $\mathcal{L}$, if $\mathcal{L}$ is abelian and if a system L' specifies an ideal in $\mathcal{L}$. Algorithms are described that compute determining systems for transporter, Lie product and Killing orthogonal subspace. This gives a systematic calculus for Lie determining systems, enabling computation of the determining systems for normalisers, centralisers, centre, derived algebra, solvable radical and key series (derived series, lower/upper central series). Our methods thereby give algorithmic access to new geometrical invariants of the symmetry action.

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[1] The first author passed away on 27 September 2015 after he made the final versions to the paper and before it was accepted for publication. This work was part of the main research with his graduate student Tracy Huang. The entire project was carried while he battled cancer.

http://dx.doi.org/10.1016/j.jsc.2016.03.002
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Please cite this article in press as: Lisle, I.G., Huang, S.-L. Tracy. Algorithmic calculus for Lie determining systems. J. Symb. Comput. (2016), http://dx.doi.org/10.1016/j.jsc.2016.03.002

1. Introduction

The symmetries of a geometric object (differential equation, exterior differential system, Riemannian metric,...) are one of its most fundamental features. They arise in various ways: to capture physical invariance properties (Bluman and Anco, 2002, §1), or as a reflection of basic mathematical properties such as linearity (Bluman et al., 2010, §2). In the study of differential equations (DEs) they give powerful methods for constructing solutions (Bluman and Anco, 2002; Olver, 1993; Ovsiannikov, 1982), and form the basis of computer algebra solution techniques (Schwarz, 2008). In computer algebra applications, the DE or other object is given, and must be analysed for its symmetries. Typically this involves 'infinitesimal' methods: if the DE or other object is on a space with coordinates $x = (x^1, \dots, x^n)$, one seeks a vector field $\xi^i(x)\frac{\partial}{\partial x^i}$ which leaves the object invariant. This gives rise to a linear homogeneous system of 'defining' or 'determining' differential equations for the symmetry vector field components $\xi^i(x)$. Examples are Lie point and contact symmetries of DEs (Bluman and Anco, 2002), 'intrinsic' or Cartan symmetries (Krasil'shchik and Vinogradov, 1999, §3.7), and Killing equations for infinitesimal isometries of a Riemannian space.

Many packages are available in computer algebra systems for finding determining systems (Carminati and Vu, 2000; Cheviakov, 2007; Rocha Filho and Figueiredo, 2011). These packages then rely on solving the system to obtain symmetry vector fields explicitly. However there is great appeal in devising methods that can infer properties of a Lie algebra directly from the determining system without solving it. There are several reasons for this. First, the determining system is the immediately available, algorithmically constructible object, whereas the *solutions* of the determining system are available only via application of integration heuristics, which inevitably fail in some instances. Second, the determining system often has coefficients that live in a computable field (rational numbers, rational functions), whereas the solutions may involve algebraic or transcendental extensions (e^x, $\sqrt{1+x^2}$ etc.) which make computer algebra manipulations clumsier and less reliable. Finally, the very notion of a 'Lie pseudogroup' of transformations is *defined* in terms of satisfaction of a determining system (Pommaret, 1978; Singer and Sternberg, 1965; Stormark, 2000); it is therefore mathematically natural to cultivate techniques which stay close to the definition and work with the system rather than its solutions. An added attraction is that finite and infinite Lie pseudogroups can be treated in a unified way.

Some previous work has exploited differential reduction and completion methods to obtain properties of a Lie algebra directly from the determining system. Schwarz (1992a) noted that the dimension of a symmetry group could be inferred directly from the system, using the classical local existence–uniqueness (E–U) theorem of Riquier (1910). Subsequently, Reid et al. (1992) exhibited a method for finding the structure constants c_{ij}^k of a Lie algebra with respect to a certain basis characterised by the determining system, without knowing the solutions. Lisle et al. (2014) extended this work to extract c_{ij}^k in the case where the determining system has been integrated fully or partially. Reid et al. (1992) suggested using their c_{ij}^k as inputs to Lie algebraic algorithms such as are described in de Graaf (2000). Application to symmetry of DE additionally requires knowledge of how the Lie group acts on space. For example, Draisma (2001) used knowledge of Reid's c_{ij}^k, plus geometric information (transitivity, isotropy algebra) available at the level of determining systems (Lisle and Reid, 1998) to give an algorithm that identifies the 'symmetry type' of an ODE of order $k \geq 2$. An alternative approach to identifying the symmetry type of ODE is taken by Schwarz (2008), again working at the level of determining systems.

From the point of view of general Lie algebras, acting on general spaces, the above work is fragmentary, being restricted for example to action on 2-dim (x, y) space. Apart from the work of Reid et al. (1992) and Lisle et al. (2014) for finding structure constants c_{ij}^k, there is no systematic calculus for dealing with determining systems of arbitrary Lie algebras of vector fields. Our purpose is to fill this gap.

This paper is one of a sequence where we develop a toolkit of algorithms for determining systems, with the goal of extracting algebraic information (isomorphism invariants) about the Lie algebra, and geometric information (diffeomorphism invariants) about the vector fields that constitute it. As well as decision procedures, for example testing whether such a system really defines a Lie algebra $\mathcal{L}$, we

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