Knowledge-Based Systems 116 (2017) 39–48

Contents lists available at ScienceDirect

Knowledge-Based Systems

journal homepage: www.elsevier.com/locate/knosys

# Constructing three-way concept lattices based on apposition and subposition of formal contexts

Ting Qian [a,b], Ling Wei [a,*], Jianjun Qi [c]

[a] School of Mathematics, Northwest University, Xi'an 710069, PR China
[b] College of Science, Xi'an Shiyou University, Xi'an 710065, PR China
[c] School of Computer Science and Technology, Xidian University, Xi'an 710071, PR China

ARTICLE INFO

*Article history:*
Received 16 February 2016
Revised 29 October 2016
Accepted 31 October 2016
Available online 2 November 2016

*Keywords:*
Three-way concept lattice
Three-way decision
Apposition
Subposition

ABSTRACT

Three-way concept analysis provides a new model to make three-way decisions. Its basic structure can be shown by the three-way concept lattices. Thus, how to construct three-way concept lattices is an important issue in the three-way concept analysis. This paper proposes approaches to create the three-way concept lattices of a given formal context. First, we can transform the given formal context and its complementary context into new formal contexts which are isomorphic to the given formal context and its complementary context respectively. And then, Type I-combinatorial context and Type II-combinatorial context are defined, which are apposition and subposition of these new formal contexts, respectively. Second, we prove that the concept lattice of Type I-combinatorial context is isomorphic to object-induced three-way concept lattice and the concept lattice of Type II-combinatorial context is isomorphic to attribute-induced three-way concept lattice of the given formal context. And then, the approaches of creating the three-way concept lattices are proposed based on the concept lattices of Type I-combinatorial context and Type I-combinatorial context. Finally, we give the corresponding algorithms of constructing three-way concept lattices based on the above approaches and conduct several experiments to illustrate the efficient of proposed algorithms.



## 1. Introduction

Based on the mathematization of concept and conceptual hierarchy, formal concept analysis (FCA), proposed by Wille in 1982 [29], is a field of applied mathematics. Its foundation includes a formal context, formal concepts and corresponding concept lattice. Most of the researches on FCA are concentrating on the following topics: construction and pruning algorithms of concept lattices [1,6,9,20,23]; acquisition of rules [13,14,19]; reduction of concept lattices [25,30,36]. Recently, studying FCA by combining it with other theories together is a hot topic in FCA research [7,8,12,17,18,21,26–28,37]. For example, Qi et al. first proposed three-way concept analysis by combining FCA with three-way decisions [21,22].

Three-way decisions are widely used in real-world decision-making. They are used in different fields and disciplines by different names and notations. Observing this phenomenon, Yao [31] proposed an outline of the theory of three-way decisions that provides an unified and discipline-independent framework for decision-making with three decisions, namely, acceptance, rejection and non-commitment. And after that many recent studies [4,10,11,15,16,21,22,24,31–35,38] investigate in various areas. For example, Yu [35] proposed a tree-based incremental overlapping clustering method by using three-way decisions.

As a new model to make three-way decisions, three-way concept analysis can be taken as a generalization of formal concept analysis. Similar to a formal concept in FCA, a three-way concept is also constituted of an extent and an intent. The difference is that the extent (or the intent) in a three-way concept is equipped with two parts: the positive one and the negative one. These two parts are used to express the semantics “jointly possessed” and “jointly not possessed” in a formal context respectively. On the basis of three-way concept, one can divide the object (or attribute) universe into three regions to make three-way decisions [21]. More specifically, “jointly possessed” means acceptance and “jointly not possessed” means rejection in the three-way decision.

The basic structure of three-way concept analysis can be shown by the three-way concept lattices. In [21] and [22], the theoretical foundations and frames of two kinds of three-way concept lattices were built and the ideology of the theory of three-way concept

* Corresponding author. Fax: +86 29 88308433.
*E-mail addresses:* qiant2000@126.com (T. Qian), wl@nwu.edu.cn (L. Wei), qijj@mail.xidian.edu.cn (J. Qi).

http://dx.doi.org/10.1016/j.knosys.2016.10.033
0950-7051/

lattices was discussed clearly. However, the authors did not consider the problem of constructing three-way concept lattices at that time.

This paper will discuss this problem and give a method to create the three-way concept lattices. As far as we know, there is one-to-one correspondence between a formal context and a complete lattice [5]. In addition, the three-way concept lattices are complete lattices. So we want to find some specific formal contexts whose concept lattices can correspond to the three-way concept lattices. If such formal contexts exist, then on one hand, the well-developed formal concept construction methods and techniques can be applied to construct three-way concept and three-way concept lattices. On the other hand, based on the relations between the new formal contexts and the original formal context, the deeper researches of the relations between formal concept lattices and three-way concept lattices can be studied.

In fact, the information of "jointly possessed" and "jointly not possessed" in a formal concept is provided by a formal context and its complementary context, respectively. Therefore, we consider to construct the above specific formal contexts by using the related formal contexts of the formal context and its complementary context. Luckily, we constructed such above mentioned formal contexts called Type I-combinatorial context and Type II-combinatorial context, and this paper will show the idea and method of constructions.

This paper is organized as follows. In Section 2, we briefly review some basic notions related to FCA and three-way concept lattices. In Section 3, some novel formal contexts and Type I-combinatorial context based on the object set are defined, and the related conclusions are given. In Section 4, the other novel formal contexts and Type II-combinatorial context based on the attribute set, are defined and some related conclusions are given, which are similar to results in Section 3. In Section 5, the corresponding algorithms of acquisition approaches to three-way concept lattices are presented. Furthermore, the related experiments are conducted. Finally, conclusions are drawn in Section 6.

## 2. Preliminaries

This section gives some necessary operators, definitions and symbols in this paper.

Some set-theoretic operators are introduced first. Let $S$ be a non-empty finite set, $\mathcal{P}(S)$ be its power set and $\mathcal{DP}(S) = \mathcal{P}(S) \times \mathcal{P}(S)$. Set-theoretic operators on $\mathcal{DP}(S)$, such as intersection $\cap$, union $\cup$ and complement $c$, are defined componentwise using standard set operators. For two pairs of subsets $(A, B), (C, D) \in \mathcal{DP}(S)$, we define $(A, B) \cap (C, D) = (A \cap C, B \cap D)$, $(A, B) \cup (C, D) = (A \cup C, B \cup D)$, $(A, B)^c = (A^c, B^c)$, and $(A, B) \subseteq (C, D) \Leftrightarrow A \subseteq C$ and $B \subseteq D$.

### 2.1. Formal concept analysis

The section reviews some basic notions and properties in FCA.

**Definition 2.1** [5]**.** A formal context $(G, M, I)$ consists of two sets $G$ and $M$ and a relation $I$ between $G$ and $M$. The elements of $G$ are called the objects and the elements of $M$ are called the attributes of the context. In order to express that an object $g$ is in a relation $I$ with an attribute $m$, we write $gIm$ or $(g, m) \in I$ and read it as "the object $g$ has the attribute $m$".

With respect to a formal context $(G, M, I)$, Wille and Ganter [5] defined a pair of dual operators for any $A \subseteq G$ and $B \subseteq M$ by:

$$A^* = \{m \in M | gIm \text{ for all } g \in A\}, B^* = \{g \in G | gIm \text{ for all } m \in B\}.$$

We say a formal context is canonical if $\forall g \in G$, $g^* \neq \emptyset$, $g^* \neq M$, and $\forall m \in M$, $m^* \neq \emptyset$, $m^* \neq G$. All the formal contexts we study in this paper are finite and canonical.

**Table 1**
A formal context $(G, M, I)$.

| $G$ | $a$ | $b$ | $c$ | $d$ | $e$ |
|---|---|---|---|---|---|
| 1 | × | × | | × | × |
| 2 | × | × | × | | |
| 3 | | | | × | |
| 4 | × | × | × | | |

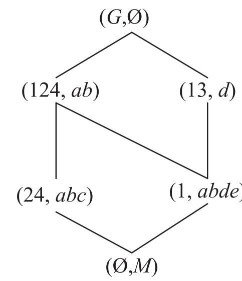


**Fig. 1.** $\underline{B}(G, M, I)$ in Example 2.1.

Let $(G, M, I)$ be a formal context. $\forall A_1, A_2, A \subseteq G$, $\forall B_1, B_2, B \subseteq M$, the following properties hold:

1. $A_1 \subseteq A_2 \Rightarrow A_2^* \subseteq A_1^*$, $B_1 \subseteq B_2 \Rightarrow B_2^* \subseteq B_1^*$.
2. $A \subseteq A^{**}$, $B \subseteq B^{**}$.
3. $A^* = A^{***}$, $B^* = B^{***}$.
4. $A \subseteq B^* \Leftrightarrow B \subseteq A^*$.
5. $(A_1 \cup A_2)^* = A_1^* \cap A_2^*$, $(B_1 \cup B_2)^* = B_1^* \cap B_2^*$.
6. $(A_1 \cap A_2)^* \supseteq A_1^* \cup A_2^*$, $(B_1 \cap B_2)^* \supseteq B_1^* \cup B_2^*$.

If $A^* = B$ and $B^* = A$, then $(A, B)$ is called a formal concept, where $A$ is called the extent of the formal concept, $B$ is called the intent of the formal concept. The family of all formal concepts of $(G, M, I)$ form a complete lattice, which is called the concept lattice and is denoted by $\underline{B}(G, M, I)$. For any $(A_1, B_1), (A_2, B_2) \in \underline{B}(G, M, I)$, the partial order is defined by:

$$(A_1, B_1) \leqslant (A_2, B_2) \Leftrightarrow A_1 \subseteq A_2 (\Leftrightarrow B_1 \supseteq B_2).$$

The infimum and supremum of $(A_1, B_1)$ and $(A_2, B_2)$ are defined by:

$$(A_1, B_1) \wedge (A_2, B_2) = (A_1 \cap A_2, (B_1 \cup B_2)^{**}),$$

$$(A_1, B_1) \vee (A_2, B_2) = ((A_1 \cup A_2)^{**}, B_1 \cap B_2),$$

respectively.

**Remark.** Since we discuss different formal context with different $I \subseteq (G \times M)$, $A^*$ and $B^*$ are denoted in the $(G, M, I)$ by $A^I$ and $B^I$ respectively in the sequel.

**Example 2.1.** A formal context $(G, M, I)$ is shown in Table 1. $G = \{1, 2, 3, 4\}$ is an object set, $M = \{a, b, c, d, e\}$ is an attribute set. The corresponding concept lattice $\underline{B}(G, M, I)$ is shown in Fig. 1, in which, every set is denoted directly by listing its elements except $G$, $M$ and $\emptyset$.

**Definition 2.2** [3]**.** Let $L$ and $K$ be lattices. A map $f$: $L \to K$ is said to be a homomorphism if $f$ is join-preserving and meet-preserving, that is, for all $a, b \in L$, $f(a \vee b) = f(a) \vee f(b)$ and $f(a \wedge b) = f(a) \wedge f(b)$. A bijective homomorphism is a lattice isomorphism.

Reference [3] also shows that $f$ is lattice isomorphism if and only if $f$ is order-isomorphism.

**Definition 2.3** [5]**.** An isomorphism between contexts $\mathbb{K}_1 = (G, M, I)$ and $\mathbb{K}_2 = (H, N, J)$ is a pair $(\alpha, \beta)$ of bijective maps $\alpha$: $G \to H$, $\beta$: $M \to N$ with $gIm \Leftrightarrow \alpha(g) J \beta(m)$.