Neural Networks 89 (2017) 50–60

Contents lists available at ScienceDirect

Neural Networks

journal homepage: www.elsevier.com/locate/neunet

Stability and instability of a neuron network with excitatory and inhibitory small-world connections

Dongyuan Yu [a], Xu Xu [a,*], Jing Zhou [a], Eric Li [b]

[a] College of Mathematics, Jilin University, 2699 Qianjin Street, Changchun 130012, PR China

[b] Department of Mechanical and Automation Engineering, The Chinese University of Hong Kong, Hong Kong, PR China

ARTICLE INFO

Article history:
Received 1 July 2016
Received in revised form 16 February 2017
Accepted 24 February 2017
Available online 2 March 2017

Keywords:
Bifurcation
Delay-dependent global stability
Matrix perturbation theory
Mean-field theory
Small-world network

ABSTRACT

This study considers a delayed neural network with excitatory and inhibitory shortcuts. The global stability of the trivial equilibrium is investigated based on Lyapunov's direct method and the delay-dependent criteria are obtained. It is shown that both the excitatory and inhibitory shortcuts decrease the stability interval, but a time delay can be employed as a global stabilizer. In addition, we analyze the bounds of the eigenvalues of the adjacent matrix using matrix perturbation theory and then obtain the generalized sufficient conditions for local stability. The possibility of small inhibitory shortcuts is helpful for maintaining stability. The mechanisms of instability, bifurcation modes, and chaos are also investigated. Compared with methods based on mean-field theory, the proposed method can guarantee the stability of the system in most cases with random events. The proposed method is effective for cases where excitatory and inhibitory shortcuts exist simultaneously in the network.

1. Introduction

In the past two decades, there has been an increasing interest in the study of complex networks regarding their mathematical modeling and real applications. The properties of complex networks are determined mainly by the way of connections between the vertices. A limiting case is the regular network with a high degree of local clustering and a large average distance; the other is the random network with negligible local clustering and a small average distance. Small-world networks are the intermediate between regular networks and random networks, where they exhibit a high degree of clustering as well as a small average distance between nodes. Common networks such as power grids, financial networks, Internet servers, forest fires, and disordered porous media behave like small-world networks (Barabasi, 2002; Boccaletti, Latora, Moreno, Chavez, & Hwang, 2006; Ganguly, Deutsch, & Mukherjee, 2009; Watts, 1999, 2004). Therefore, there have been numerous studies on the small-networks since the pioneering study of Watts and Strogatz (1998).

In recent years, it has been shown that the nervous system generally exhibits small-world properties. For example, many large-scale neural networks in the brain, such as the visual system and brain stem, exhibit small-world properties (Humphries, Gurney, & Prescott, 2006). The anatomical connections in the brain and the synchronization networks of cortical neurons exhibit small-world topologies (Shan, Huang, Singer, & Nikolić, 2008; Sporns, Chialvo, Kaiser, & Hilgetag, 2004). Humphries et al. suggested that the brain stem reticular structure can be portrayed as a neural network with a small-world topology (Humphries et al., 2006). Functional magnetic resonance imaging experiments have also shown that the cerebral cortex and regions throughout the cortex exhibit small-world properties (Salvador et al., 2005).

In recent years, many studies have considered the complicated dynamics of neural networks with small-world connections (Bao, Park, & Cao, 2016; Cao & Wan, 2014; Gray & Robinson, 2009; Gu, 2008; Gu, Fang, & Wang, 2013; Horikawa, 2014; Horikawa & Kitajima, 2009; Li & Chen, 2004; Mao & Hu, 2010; Wang, Shi, & Chen, 2011; Xu, Hu, & Wang, 2006; Xu, Luo, & Gu, 2012; Xu & Wang, 2009; Yang, 2001), where one of the main problems is the stability of the equilibrium. For example, when neural networks are employed in parallel computation and signal processing to solve optimization problems, there needs to be a well-defined computable solution for all possible initial states. From the viewpoint of mathematics, this means that the network should have a unique globally stable equilibrium (Xu, 2008). Recently, the stability and instability of a delayed small-world network with excitatory *or* inhibitory shortcuts were investigated using matrix perturbation theory (Zhou, Xu, Yu, & Zheng, 2016). The stability of a neural network with small-world

* Corresponding author.
E-mail address: xuxu@jlu.edu.cn (X. Xu).

http://dx.doi.org/10.1016/j.neunet.2017.02.009
0893-6080/

connections was also analyzed using mean-field theory (Li & Chen, 2003). A dynamical model of the total volume of virus influence was constructed for a small-world network and the effects of multiple delays on the network dynamics were investigated (Xu & Luo, 2013). A large-scale brain network with the small-world characteristic of dense local clustering between neighboring populations was studied (Gray, Fung, & Robinson, 2009). A detailed analysis of the dynamics of a ring network with shortcuts was presented, such as synchronization, stability, and Hopf bifurcation (Man, Guo, & He, 2012). A system comprising a pair of networks connected to each other by one link was investigated, where each network had a unidirectional ring and a shortcut (Cheng, 2009). The dynamical behavior of a complex network with four neurons and two shortcuts with multiple delays was considered (Kundu, Das, & Roy, 2013).

Most of these studies focused on systems with only fixed excitatory shortcuts, which implies that the edges between the nodes are either present or not, and the connection weights between nodes are either 1 or 0. However, many networks have a mixture of excitatory and inhibitory connections in the real world, such as brain networks, neural networks, and interpersonal networks (Buice & Chow, 2013; Gleeson, Melnik, Ward, Porter, & Mucha, 2012). For example, your contacts may be your friends or competitors. A friend is assumed to be a positive (or excitatory) connection and a competitor is assumed to be a negative (or inhibitory) connection. Thus, it is more practical to describe a neural network model with the excitatory and inhibitory connections, but little progress has been achieved on this topic. Indeed, it is not clear whether the neural networks with excitatory and inhibitory shortcuts are easier to stabilize compared with those with regular connections. This motivated us to investigate the dynamical behaviors of these systems.

In addition to shortcuts, time delays also affect the dynamics of complex network systems. Recently, many studies have investigated the global stability of neural network models, but most focused on delay-independent stable criteria. In this case, a delay can only affect the convergence rate; however in some cases, delays play an important role when analyzing whether a system is globally stable or not. In terms of global stability, the key problem is constructing an appropriate Lyapunov function. Regarding local stability, the key problem is finding the eigenvalues of the adjacent matrix of the network. However, it is not easy to obtain the exact distribution of eigenvalues because there may be a large number of connection modes due to the randomness of shortcuts. Recently, mean-field theory has been used to analyze the stability of small-world networks (Grabow, Grosskinsky, & Timme, 2012; Li & Chen, 2003); however it cannot be employed to analyze the cases where both excitatory and inhibitory connections exist simultaneously.

In this study, we investigate the global stability of a system based on Lyapunov's direct method and the local stability is then analyzed using matrix perturbation theory. The aim of this study is to determine the stability criterion, and then analyze the mechanism of instability and the bifurcation modes. The remainder of this paper is organized as follows. In Section 2, we describe the mathematical model of a small-world network. In Section 3, we construct an energy function to give the criteria for global stability. In Section 4, the generalized sufficient conditions for the local stability are analyzed based on matrix perturbation theory. In Section 5, we compare the stability determined using the proposed method with the mean-field method. The results of some numerical experiments are presented in Section 6. Finally, we give the conclusions in Section 7.

2. Network model

In this study, a small-world network Newman–Watts model (Watts, 2004) is considered, as shown in Fig. 1. We start with a ring network of size n, where each node is connected to its left- and right-hand side $k/2$ nearest-neighbors, where k is even and $k \ll n$. Shortcuts are then added between two randomly chosen long-range nodes with a small excitatory possibility p_+, or inhibitory possibility p_-. No multiple connections or self-connections are allowed in this model.

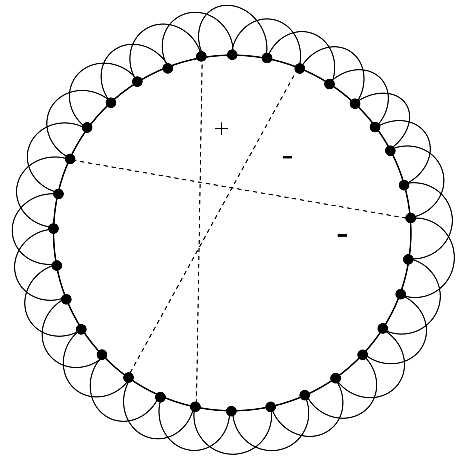

Fig. 1. Schematic showing the network with small-world connections.

The dynamical equations are described as follows:

$$\dot{x}_i(t) \equiv \frac{\mathrm{d}x_i}{\mathrm{d}t} = -x_i(t) + \sum_{j=i-k/2, j\neq i}^{i+k/2} \beta Q_{ij} f[x_j(t-\tau)] + \sum_{j=1}^{n} \beta R_{ij} f[x_j(t-\tau)], \tag{1}$$

where x_i ($i = 1, 2, \dots, n$) is the state of the ith neuron, index i is taken to modulo n, the time delay $\tau \geq 0$ is due to the finite transfer speed of the signal (Stépán, 1989; Wang & Hu, 2002); and the functions $f(x)$ and $f'(x) \equiv \mathrm{d}f(x)/\mathrm{d}x$ are assumed to be bounded piecewise monotone and continuous functions with $f(0) = 0$. It is well known that $f(x) = \tanh(x)$ is the most commonly used activation function in neural networks, and thus it is used as the activated function in this paper for simplicity. Q_{ij} and R_{ij} are the connections of regular connections and the shortcuts between the ith and the jth neuron, respectively, and $\beta > 0$ is the connection strength.

For simplicity, we investigate an undirected network, i.e.: $R_{ij} = R_{ji}$, $Q_{ij} = Q_{ji}$, and

$$Q_{ij} = \begin{cases} 1, & \text{for } j = i - \frac{k}{2} \leq j \leq i + \frac{k}{2} \\ & j \neq i \\ 0, & \text{otherwise}, \end{cases} \qquad R_{ij} = \begin{cases} +1, & \text{for } p_+ \\ -1 & \text{for } p_- \\ 0, & \text{no shortcuts.} \end{cases} \tag{2}$$

Then, Eq. (1) can be written as

$$\dot{x}_i(t) = -x_i(t) + \sum_{j=1}^{n} b_{ij} f(x_j(t-\tau)), \tag{3}$$

where $i = 1, 2, \dots, n$, $b_{ij} = \beta(Q_{ij} + R_{ij})$ with $b_{ij} = b_{ji}$ and $b_{ii} = 0$.

3. Delay-dependent global asymptotic stability

Recently, many studies have investigated the global stability of Eq. (3), but most focused on the delay-independent stable criteria. However, the global stability of the system is strongly related to the