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Stabilization of metastable dynamical rotating waves in a ring of unidirectionally coupled sigmoidal neurons due to shortcuts

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HIGHLIGHTS

- Effects of a shortcut on rotating waves in a ring of sigmoidal neurons are considered.
- A kinematical equation for the propagation of wave fronts in a rotating wave is derived.
- Rotating waves can be stabilized in the presence of an inhibitory shortcut.
- Dynamical metastability of rotating waves is lost in the presence of an excitatory shortcut.

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ABSTRACT

Effects of shortcut connection on metastable dynamical rotating waves in a ring of sigmoidal neurons with unidirectional excitatory coupling are considered. A kinematical equation describing the propagation of wave fronts is derived with a sign function for the output function of neurons. Unstable rotating waves can be stabilized in the presence of an inhibitory shortcut. When a shortcut is excitatory and connects the most distant neurons, the dynamical metastability of rotating waves is lost. The duration of transient rotating waves then increases only linearly with the number of neurons, not exponentially. However, the dynamical metastability of rotating waves remains when a shortcut is local.

1. Introduction

Metastable dynamical transient patterns have been observed in bistable arrays of coupled neuron models recently (Horikawa, 2011, 2013, 2014; Horikawa & Kitajima, 2009a, 2009b). Transient rotating waves or transient spatially nonuniform patterns last for an extremely long time until the array converges to one of stable spatially uniform steady states eventually. Their duration increases exponentially with the number of neurons so that the array never converges to its asymptotically stable state within a practical time when the number of neurons is large. Then, such metastable dynamical transient states can play more important roles than asymptotically stable states.

Such dynamical metastability comes from the symmetric bistability of the systems. It has been found in a reaction–diffusion equation with symmetric cubic nonlinearity (Bronsard & Kohn, 1990; Carr & Pego, 1989; Ei & Ohta, 1994; Fusco & Hale, 1989; Kawasaki & Ohta, 1982), which is referred to as the time-dependent Ginzburg–Landau equation, the Allen–Cahn equation or the Schlögl model in the field of phase transition. The kinematics of kinks and anti-kinks in a one-dimensional space has shown that their movement is exponentially slow with their distance. Such metastable dynamics of kinks and anti-kinks has also been examined in a spatially discrete version of the reaction–diffusion equation (Chow, Mallet-Paret, & Van Vleck, 1997; Estep, 1994; Grant & Van Vleck, 1995). Further, it has been shown that dynamical metastability emerges in several kinds of reaction–diffusion–convection equations and other systems through singular perturbation (Ward, 1996, 1998, 2001).

Studies on a ring of coupled sigmoidal neurons originates from Amari (1978), in which it was shown that a ring of unidirectionally coupled neurons can cause a stable oscillation when the number of inhibitory couplings is odd. The oscillation is a rotating wave propagating in a ring and it is qualitatively the same as a ring oscillator, which is a circular array of unidirectionally coupled odd numbers of inverters. Further, one-dimensional and two-dimensional arrays of piecewise linear sigmoidal neurons are referred to as cellular neural networks (CNN) and have been applied to signal processing and pattern recognition as a kind

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of artificial neural networks (Chua & Yang, 1988; Crounse, Chua, Thiran, & Setti, 1996; Setti, Thiran, & Serpico, 1998; Thiran, 1997; Thiran, Crounse, Chua, & Hasler, 1995; Thiran, Setti, & Hasler, 1998). Effects of delays have also been widely studied on rings of sigmoidal neurons with unidirectional and bidirectional coupling (see references in Horikawa, 2014).

Shortcut connection (nonlocal coupling, small-world connection) in networks has attracted much attention since the notions of complex networks and small-world networks were proposed (Albert & Barabasi, 2002; Barabashi & Albert, 1999; Boccaletti, Latora, Moreno, Chavez, & Hwanga, 2006; Strogatz, 2001; Watts & Strogatz, 1998). Topological and dynamical complexity has been examined in information and communication networks as well as biological, ecological and social networks. Since studies on shortcut connection have ranged over various kinds of networks in various fields, we avoid citing literature here. Concerning a ring of coupled sigmoidal neurons, some studies have been carried out on the effects of shortcuts in the presence of delays. Since a ring of coupled neurons acts as a neural oscillator and other functional units as mentioned above, the effects of shortcuts on its behaviors are of great importance. The existence, stability and bifurcations of equilibria and periodic solutions in ring of unidirectionally and bidirectionally coupled four neurons with shortcuts and delays have been studied (Kundu, Das, & Roy, 2013; Man, Guo, & He, 2012; Mao & Hu, 2008, 2009a, 2009b). A ring of unidirectionally coupled N neurons with a shortcut and delays has also been examined (Xu & Wang, 2009). It has then been shown that the existence of shortcuts causes qualitative changes in the behaviors of a ring of coupled neurons. In the results of these studies, however, delays are necessary for the stabilization of periodic solutions and the emergence of chaotic oscillations due to shortcuts. To the best of the author's knowledge, the effects of shortcut connection on rotating waves in a ring of unidirectionally coupled sigmoidal neurons without delays have not been studied. We will show that rotating waves and periodic solutions in the ring can be stabilized owing to shortcut connection even though there are no delays.

In this paper, we consider the effects of shortcut connection on the metastable dynamical rotating waves in a ring of unidirectionally coupled sigmoidal neurons without delays. We assume that all neurons are identical and all coupling strengths are the same except for shortcut connection. First, we deal with a global shortcut which connects two neurons apart from each other by a half of the number of neurons in a ring. In this network, an unstable periodic rotating wave is generated from the origin through the Hopf bifurcation as the output gain of neurons increases if the strength of a shortcut is not large. A generated rotating wave is symmetric, i.e. neurons are divided into two bumps with the same size, each including a half of neurons. The states of neurons in one bump are positive while those in the other bump is negative, and fronts (boundaries) between the bumps rotate in the ring in the direction of coupling. Then, a kinematical equation describing the propagation of wave fronts in a rotating wave is derived with a sign function for the output function of neurons. It is then shown that the rotating wave is stabilized as the strength of inhibitory (negative) shortcut connection increases. It is also shown that the duration of transient rotating waves increases only linearly with the number of neurons. Thus, the metastable dynamics of transient rotating waves, the duration of which increases exponentially with the number of neurons, is lost.

Next, we deal with a local shortcut which connects two neurons apart from each other by less than a half of the number of neurons. It is shown that an exponential increase in the duration with the number of neurons remains for rotating waves with the bums longer than the length of a shortcut. Further, a forward shortcut (the same direction as unidirectional coupling) has less effects on metastable dynamics than a backward shortcut (the opposite direction to unidirectional coupling).

The rest of the paper is organized as follows. In Section 2, a model is presented and the bifurcations of rotating waves in it are shown. Then, the kinematical description for the propagation of a wave front is explained in Section 3. In Section 4, a ring of unidirectionally coupled neurons with a global shortcut is considered. It is shown that rotating waves are stabilized in the presence of inhibitory shortcut connection and that the duration of transient rotating waves decreases in the presence of excitatory shortcut connection. In Section 5, a unidirectionally coupled ring with a local shortcut is considered. It is shown that the dynamical metastability of rotating waves remains even in the presence of excitatory shortcut connection. Discussion and conclusion are given in Sections 6 and 7, respectively. Other miscellaneous kinematics of a wave front is shown in the Appendix.

2. A model and the bifurcations of rotating waves

2.1. A model and rotating waves

We consider the following ring of unidirectionally coupled sigmoidal neurons with shortcut connection.

$$
\begin{aligned}
&\mathrm{d}x_n/\mathrm{d}t = -x_n + f(gx_{n-1}) + \delta_{n0}c_1 f(gx_K) + \delta_{nK}c_2 f(gx_0)\\
&f(x) = \tanh(x), \qquad \delta_{nn'} = \begin{cases} 1 & (n = n') \\ 0 & (n \neq n') \end{cases}\\
&(0 \le n \le N-1,\ 2 \le K \le N/2\ (\equiv l_h),\ x_{n\pm N} = x_n,\ g > 0)
\end{aligned}
\tag{1}
$$

where x_n is the state of the nth neuron, $f(x)$ is the sigmoidal output function of neurons and g is an output gain. Each neuron connects unidirectionally to the next neuron with unit coupling strength, and a total of N neurons make a circular array. We let N be even ($N = 2l_h$) for the sake of simplicity, but the obtained results are applicable to a ring of an odd number of neurons. In addition, the zeroth and Kth neurons connect to each other with the coupling strength c_1 and c_2, which forms shortcut connection. The length of a shortcut is K and we let $K \le l_h$ ($= N/2$) without the loss of generality. We refer to a shortcut with the length $K < l_h$ as a global shortcut and a shortcut with the length $K < l_h$ as a local shortcut. We also refer to a shortcut from x_K to x_0 as a backward shortcut and refer to a shortcut from x_0 to x_K as a forward shortcut for a local shortcut with $K \le l_h - 1$ according to the direction of unidirectional coupling and the propagation of a rotating wave.

Eq. (1) is symmetric with respect to changes in the signs of neurons, i.e., if x_n ($0 \le n \le N-1$) is a solution to Eq. (1), then $-x_n$ ($0 \le n \le N-1$) is also a solution to Eq. (1). The origin ($x_n = 0, 0 \le n \le N-1$) is always a steady state and the Jacobian matrix J evaluated at the origin is given by Eq. (2).

$$
\begin{array}{cc}
J = \left[\begin{array}{cccccccc}
-1 & 0 & \cdot & 0 & c_1g & \cdot & 0 & g\\
g & -1 & \cdot & 0 & 0 & \cdot & 0 & 0\\
0 & g & \cdot & 0 & 0 & \cdot & 0 & 0\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\
c_2g & 0 & \cdot & g & -1 & \cdot & 0 & 0\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\
0 & 0 & \cdot & 0 & 0 & \cdot & -1 & 0\\
0 & 0 & \cdot & 0 & 0 & \cdot & g & -1
\end{array}\right] &
\begin{array}{c}
\text{row}\\ 0\\ 1\\ 2\\ \cdot\\ K\\ \cdot\\ N-2\\ N-1
\end{array}\\
\begin{array}{cccccccc}
\text{column} & 0 & 1 & K-1 & K & N-2 & N-1
\end{array} &
\end{array}
\tag{2}
$$

In the absence of shortcut connection ($c_1 = c_2 = 0$), the Jacobian matrix equation (2) is a simple circulant matrix. The eigenvalues λ_n^0 ($0 \le n \le N-1$) of Eq. (2) are given by solutions to the characteristic equation $|J - \lambda_n I| = (-1-\lambda_n)^N + (-1)^{N-1}g^N = 0$