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Boundedness and periodicity for linear threshold discrete-time quaternion-valued neural network with time-delays☆

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ABSTRACT

In this paper, the discrete-time quaternion-valued neural network with linear threshold activation functions is investigated. The sufficient conditions to the boundedness and global exponential periodicity of the neural network are obtained by using characteristic equation, Lyapunov functional and *M*-matrix. Simulation results illustrative the effectiveness of the conclusions obtained in this paper.

1. Introduction

Processing multi-dimensional data is an important problem for artificial neural networks. The simplest form of multi-dimensional neural networks is complex-valued neural networks (CVNNs in short) where the complex number system is utilized to represent two-dimensional data elements as a single entity. CVNNs play an important role in various engineering applications involving complex-valued data, such as adaptive signal processing, communication engineering and quantum mechanics, see [3,6,13,17,18,20,21,41] and the references therein. However for higher-dimensional data, we need a number system with higher dimensions, the so-called hypercomplex number systems.

The quaternion, first described by Irish mathematician William Rowan Hamilton in 1843, is a four-dimensional hypercomplex number system that extends the complex numbers. Quaternions have great potential in three- and four-dimensional data modeling and have found application in the areas of engineering, including computer graphics [7,8,29] and robotics [9]. Representing high-dimensional information, such as color and three-dimensional body coordinates, by quaternion-valued neurons are more efficient than complex-valued neurons or real-valued neurons. Thus the quaternion-valued neural network (QVNN in short) models, with quaternion-valued states, connections weights and activation functions were first proposed in [35] and studied in recent years, see [24,26,27,36].

QVNNs are much more complicated than CVNNs in the analysis process. First difficulty is the non-commutativity of quaternion multiplication. The Hamilton rules implies that the quaternion multiplication is non-commutative. We will give the details in Section 2. Second, the analyticity in quaternion field $\mathbb{H}$ is governed by the Cauchy–Riemann–Fueter (CRF) conditions and the generalized Cauchy–Riemann (GCR) conditions [14,32]. According to these two conditions, globally analytical quaternion-valued functions are only linear functions and constants. Therefore, choosing appropriate quaternion-valued activation functions of QVNNs remains an challenge. To partially overcome this issue, a suboptimal solution in the form of a split nonlinear quaternion function that treats each channel separately (as a real channel) passed through a real smooth nonlinearity was employed in [1].

In recent decades, neural networks with linear threshold activation functions have found various applications in associative memory, winner-take-all and group selection, see [15,30,33,34]. The linear threshold functions are unsaturating piecewise linear functions that are more biologically plausible than sigmoid and limiter functions, see [10,11,19]. Compared with neural networks

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with nonlinear activation functions, one advantage of the neural networks with linear threshold activation functions is that the network can be viewed as a linear system if each neuron's output is always greater than or less than zero, and the neural networks with linear threshold activation functions will change one linear system to another similar linear system only when some neurons' outputs switch on the zero boundary.

The dynamics of recurrent neural networks, including boundedness, stability, periodicity and synchronization, are frequently studied in recent years, see [5,12,13,16,23,25,37,41] etc. When implementing the continuous-time neural networks for simulation, experimentation or computation, it is essential to construct a discrete-time neural network which is an analogue of the continuous-time neural network. Some researchers have discussed the significance for discrete-time analogues to reflect the dynamics of their continuous-time counterparts [28,31]. It is usually expected that the discrete-time analogue inherit the dynamical characteristics of the continuous-time delayed neural networks. Although there are numerous ways of obtaining discrete-time neural networks from continuous-time systems, most of the discrete-time neural networks do not faithfully preserve the dynamics of their continuous-time versions. As pointed out in [22], the discretization cannot preserve the dynamics of the continuous-time counterpart even for a small sampling period, and therefore there is a crucial need to study the dynamics of discrete-time neural networks. The dynamics of both real-valued neural networks and CVNNs with linear threshold activation functions have been investigated in [13,38–40,42,43]. However, to the best of our knowledge, there are few works on the dynamical analysis on discrete-time QVNNs with linear threshold activation functions.

Motivated by the above discussion, in this paper we aim to study the boundedness and periodicity for discrete-time linear threshold quaternion-valued neural network with time-delays. The rest of the papers is organized as follows. In Section 2, the quaternion algebra is introduced as well as the model description and some useful definitions and lemmas. In Section 3, the sufficient conditions for the boundedness and periodicity of the proposed QVNN models are obtained. In Section 4, we provide two numerical examples to illustrate the effectiveness of the results. Conclusions are given in Section 5.

2. Preliminaries

2.1. Quaternion algebra

Quaternions are an associative algebra defined over $\mathbb{R}$, where the quaternion number q is given by

$$q = q^{\mathrm{R}} + q^{\mathrm{I}}\imath + q^{\mathrm{J}}\jmath + q^{\mathrm{K}}\kappa$$

where $q^{\mathrm{R}}, q^{\mathrm{I}}, q^{\mathrm{J}}, q^{\mathrm{K}} \in \mathbb{R}$ and ı, ȷ, κ are both orthogonal unit vectors and imaginary units. The imaginary units ı, ȷ, κ obey the following Hamilton rules

$$\begin{cases} \imath\jmath = \kappa; \jmath\kappa = \imath; \kappa\imath = \jmath; \\ \imath^2 = \jmath^2 = \kappa^2 = \imath\jmath\kappa = -1. \end{cases} \tag{1}$$

which implies the non-commutativity of the quaternion multiplication.

The operations between quaternions, $p = p^{\mathrm{R}} + p^{\mathrm{I}}\imath + p^{\mathrm{J}}\jmath + p^{\mathrm{K}}\kappa$ and $q = q^{\mathrm{R}} + q^{\mathrm{I}}\imath + q^{\mathrm{J}}\jmath + q^{\mathrm{K}}\kappa$, are defined as follows. The addition and subtraction of quaternions are defined similarly as those of complex numbers, by

$$p \pm q = (p^{\mathrm{R}} \pm q^{\mathrm{R}}) + (p^{\mathrm{I}} \pm q^{\mathrm{I}})\imath + (p^{\mathrm{J}} \pm q^{\mathrm{J}})\jmath + (p^{\mathrm{K}} \pm q^{\mathrm{K}})\kappa$$

According to Hamilton multiplication rules (1), the product of p and q is defined as

$$\begin{aligned} pq = & (p^{\mathrm{R}}q^{\mathrm{R}} - p^{\mathrm{I}}q^{\mathrm{I}} - p^{\mathrm{J}}q^{\mathrm{J}} - p^{\mathrm{K}}q^{\mathrm{K}}) \\ & + (p^{\mathrm{R}}q^{\mathrm{I}} + p^{\mathrm{I}}q^{\mathrm{R}} + p^{\mathrm{J}}q^{\mathrm{K}} - p^{\mathrm{K}}q^{\mathrm{J}})\imath \\ & + (p^{\mathrm{R}}q^{\mathrm{J}} + p^{\mathrm{J}}q^{\mathrm{R}} - p^{\mathrm{I}}q^{\mathrm{K}} + p6\mathrm{K}q^{\mathrm{I}})\jmath \\ & + (p^{\mathrm{R}}q^{\mathrm{K}} + p^{\mathrm{K}}q^{\mathrm{I}} + p^{\mathrm{I}}q^{\mathrm{J}} - p^{\mathrm{J}}q^{\mathrm{I}})\kappa. \end{aligned}$$

The module for a quaternion $q = q^{\mathrm{R}} + \imath q^{\mathrm{I}} + \jmath q^{\mathrm{J}} + \kappa q^{\mathrm{K}} \in \mathbb{H}$, denoted by $|q|$, is defined as

$$|q| = \sqrt{(q^{\mathrm{R}})^2 + (q^{\mathrm{I}})^2 + (q^{\mathrm{J}})^2 + (q^{\mathrm{K}})^2}.$$

In this paper, the vector norm $||q||_1$ and $||q||_\infty$ (simply denoted as $||q||$) of a quaternion vector $q = (q_1, q_2, \ldots, q_n)^{\mathrm{T}} \in \mathbb{H}^n$ is defined as

$$||q||_1 = \sum_{i=1}^{n}(|q_i^{\mathrm{R}}| + |q_i^{\mathrm{I}}| + |q_i^{\mathrm{J}}| + |q_i^{\mathrm{K}}|), \; ||q||_\infty = \max_{1\le i\le n}\{|q_i|\}.$$

2.2. Model description

In this paper, we investigate a class of discrete-time linear threshold QVNNs with time delays described by the following nonlinear discrete equation

$$q(k+1) = -Dq(k) + Af(q(k)) + Bf(q(k-\tau)) + u(k) \tag{2}$$

where $q(k) = (q_1(k), q_2(k), \ldots, q_n(k))^{\mathrm{T}} \in \mathbb{H}^n$ is the state vector, $D = \mathrm{diag}(d_1, d_2, \ldots, d_n) \in \mathbb{R}^{n\times n}$ with $d_i > 0$ $(i = 1, 2, \ldots, n)$ is the self-feedback connection weight matrix, $A = (a_{ij})_{n\times n} \in \mathbb{H}^{n\times n}$ and $B = (b_{ij})_{n\times n} \in \mathbb{H}^{n\times n}$ are, respectively, the connection weight matrix without and with time delays, $f(q(k)) = (\tilde{f}(q_1(k)), \tilde{f}(q_2(k)), \ldots, \tilde{f}(q_n(k)))^{\mathrm{T}} : \mathbb{H}^n \to \mathbb{H}^n$ is a quaternion-valued activation function where $\tilde{f}(\cdot) : \mathbb{H} \to \mathbb{H}$ is defined as

$$\tilde{f}(q) = \sigma(q^{\mathrm{R}}) + \sigma(q^{\mathrm{I}})\imath + \sigma(q^{\mathrm{J}})\jmath + \sigma(q^{\mathrm{K}})\kappa, \quad q \in \mathbb{H} \tag{3}$$

and since, as stated in the introduction, the linear threshold functions $\sigma(\cdot) : \mathbb{R} \to \mathbb{R}$ are unsaturating piecewise linear functions, $\sigma(\cdot)$ is usually defined as (see [13,38,43], etc.)

$$\sigma(x) = \max\{0, x\}, \quad x \in \mathbb{R} \tag{4}$$

and $\tau > 0$ is the constant time delay, $u(k) = (u_1(k), u_2(k), \ldots, u_n(k))^{\mathrm{T}} \in \mathbb{H}^n$ is the external input vector-valued function with period ω.

Definition 1. The neural network (2) is said to be bounded if each of its trajectories is bounded.

Definition 2. The state vector $q(k)$ of QVNN (2) is said to be globally exponentially stable at a period orbit $\hat{q}(k; k_0, \hat{\phi})$ if there exist constants $\alpha > 0$ and $\beta > 0$ such that $\forall k \ge k_0$,

$$||q(k; k_0, \phi) - \hat{q}(k; k_0, \hat{\phi})|| \le \beta||\phi - \hat{\phi}||_{k_0} \exp\{-\alpha(k - k_0)\}$$

where $q(k; k_0, \phi)$ is the state of (2) with an arbitrary initial condition ϕ and $\hat{q}(k; k_0, \hat{\phi})$ is an orbit of (2) with a certain initial condition $\hat{\phi}$, $||\cdot||_{k_0}$ is some appropriate quaternion module.

Lemma 1. *For any quaternion $q \in \mathbb{H}$ and function $\tilde{f}(q)$ defined in (3), we have*

$$|\tilde{f}(q)| \le |q|.$$

Proof. Obviously, for $\sigma(\cdot)$ defined in (4), we have

$$|\sigma(x)| = |\max\{0, x\}| \le |x|, \quad x \in \mathbb{R}.$$

then we have