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Zeroing neural networks: A survey☆

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ABSTRACT

Using neural networks to handle intractability problems and solve complex computation equations is becoming common practices in academia and industry. It has been shown that, although complicated, these problems can be formulated as a set of equations and the key is to find the zeros of them. Zeroing neural networks (ZNN), as a class of neural networks particularly dedicated to find zeros of equations, have played an indispensable role in the online solution of time-varying problem in the past years and many fruitful research outcomes have been reported in the literatures. The aim of this paper is to provide a comprehensive survey of the research on ZNNs, including continuous-time and discrete-time ZNN models for various problems solving as well as their applications in motion planning and control of redundant manipulators, tracking control of chaotic systems, or even populations control in mathematical biosciences. By considering the fact that real-time performance is highly demanded for time-varying problems in practice, stability and convergence analyses of different continuous-time ZNN models are reviewed in detail in a unified way. For the case of discrete-time problems solving, the procedures on how to discretize a continuous-time ZNN model and the techniques on how to obtain an accuracy solution are summarized. Concluding remarks and future directions of ZNN are pointed out and discussed.

1. Introduction

Approaches based on neural network for solving various knotty problems have attracted considerable attention in many fields [1–14]. For example, an adaptive fuzzy controller based on neural network is constructed for a class of nonlinear discrete-time systems with discrete-time dead zone in [1]. An adaptive decentralized scheme based on neural network is presented for multiple-input and multiple-output (MIMO) nonlinear systems with the aid of back-stepping techniques in [14]. Such a scheme guarantees the uniform ultimate boundedness of all signals in the closed-loop system with respect to mean square. To overcome the design difficulty of nonstrict-feedback structure, Ref. [3] uses variable separation technique to decompose the unknown functions of all state variables into a sum of smooth functions of each error dynamic. With the aid of radial basis function neural networks' universal approximation capability, an adaptive neural control algorithm is proposed in [3]. Authors in [8] propose a neural network model to generate winner-take-all competition, which has an explicit explanation of the competition mechanism. As a branch of artificial intelligence, recurrent neural network (RNN) models have received considerable investigation in many scientific and engineering fields, which is often exploited for computational problems [15–22] and nonlinear optimizations are solved by many methods [23,24] A gradient-based RNN model is presented in [25] for computing the inversion of a matrix online with guaranteed convergence, which can be deemed as a seminal work in this field. A simplified neural network model is presented in [26] to solve a class of linear matrix inequality problems, of which the stability and solvability are analyzed theoretically. In general, recurrent neural networks can be divided into two classes: (1) the continuous-time RNNs and (2) the discrete-time RNNs. By exploiting a numerical differential formula, a continuous-time RNN model can be discretized into a discrete-time one. However, a numerical differentiation rule does not necessarily generate a conver-

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gent and stable discrete-time RNN model even though the original continuous-time RNN model is convergent. In addition, if the discrete-time RNN model is coded as a serial-processing program and performed on the digital computer, it can be considered as a numerical algorithm [27]. As a novel type of RNN specifically designed for solving time-varying problems, zeroing neural network (ZNN) is able to perfectly track time-varying solution by exploiting the time derivative of time-varying parameters. Then, many researchers make progresses along this direction by proposing various kinds of ZNN models for solving problems with different highlights. A detailed survey and summary are necessary for understanding the development of ZNN models as well as their applications. This paper is organized as follows. In Section 2, the descriptions and continuous-time ZNN models are presented, which include the evolution of models, activation functions, finite-time convergence, and integration-enhanced ZNN models. In Section 3, a brief review on the discrete-time ZNN models is presented. In Section 4, the applications of ZNN techniques are also analyzed. Section 5 concludes this paper with finial remarks.

2. Design formulas and various continuous-time models

Prior to the proposal of ZNN approach, many gradient-related methods had been reported on the solutions of algebraic equations and optimizations, i.e., zero-finding problems [28–31]. By constructing a performance index whose minimal point is identical to the solution to the task problem, a typical approach is to design a recurrent neural network evolving along the negative gradient descent to achieve a minimum of the performance index. However, these methods may fail to work well when exploited to the online solution of dynamic problems with time-varying coefficients, which is intrinsically due to the lacking of the compensation to the velocity components of the time-varying coefficients. Therefore, in view of the variability of coefficients, any method designed intrinsically for computing the static problem can no longer guarantee the decrease of the performance index of a time-varying problem, thereby possibly leading to a failure of the task with large residual error. For example, it is observed, investigated and analyzed in [32–34] that the residual error of gradient-based neural network (GNN) for solving a time-varying problem can not be eliminated and remains at a relative high level. Refs. [27,35,36] further point out that, when exploited to solve a time-varying problem, any traditional method that does not exploit the time-derivative information of time-varying coefficients can not converge to the theoretic solution with the residual error proportional to the value of the sampling gap.

To solve a time-varying problem in an error-free manner, Zhang et al. present a recurrent neural network for solving the time-varying Sylvester equation, which is depicted in an implicit dynamical system and can be deemed as the seminal work on ZNN [37]. They further generalize and summarize the design procedures of such a methodology, and analyze the convergence and stability of the corresponding ZNN model for time-varying matrix inversion in [38]. Specifically, for solving a time-varying matrix inversion problem depicted in the form of

$$A(t)X(t) = I, \tag{1}$$

where $A(t) \in \mathbb{R}^{(n\times n)}$ is a smooth matrix with its derivative assumed to be known, $I \in \mathbb{R}^{(n\times n)}$ is the identity matrix, $X(t) \in \mathbb{R}^{(n\times n)}$ is the unknown matrix to be obtained.

The core in the design of ZNN model is to construct an error function $E(t) = A(t)X(t) - I$, which is evidently different from the performance index of gradient-related methods. Then, the ZNN design formula is used to enforce the corresponding $E(t)$ to converge to zero:

$$\dot{E}(t) = -\gamma\Phi(E(t)), \tag{2}$$

Table 1
Continuous-time ZNN models constructed for solving time-varying problems.

	Dynamic problem	Error function	ZNN model
[41]	4th root finding $x^4(t) = a(t)$	$e(t) = x^4(t) - a(t)$	$\dot{x}(t) = \frac{\dot{a}(t)-\gamma\phi(x^4(t)-a(t))}{4x^3(t)}$
[44]	Linear system $A(t)\mathbf{x}(t) = \mathbf{b}(t)$	$\mathbf{e}(t) = A(t)\mathbf{x}(t) - \mathbf{b}(t)$	$A(t)\dot{\mathbf{x}}(t) = -\dot{A}(t)\mathbf{x}(t) + \dot{\mathbf{b}}(t) - \gamma\Phi(A(t)\mathbf{x}(t) - \mathbf{b}(t))$
[38]	Matrix inversion $A(t)X(t) = I$	$E(t) = A(t)X(t) - I$	$A(t)\dot{X}(t) = -\dot{A}(t)X(t) - \gamma\Phi(A(t)X(t) - I)$
[45]	Matrix square roots finding $X^2(t) = A(t)$	$E(t) = X^2(t) - A(t)$	$X(t)\dot{X}(t) + \dot{X}(t)X(t) = -\gamma\Phi(X^2(t) - A(t)) - \dot{A}(t)$
[46]	Nonlinear equations $\mathbf{f}(\mathbf{x}(t), t) = 0$	$\mathbf{e}(t) = \mathbf{f}(\mathbf{x}(t), t)$	$\dot{\mathbf{x}}(t) = -J^{-1}(\mathbf{x}(t), t)\left(\gamma\Phi(\mathbf{f}(\mathbf{x}(t), t)) + \frac{\partial \mathbf{f}(\mathbf{x}(t),t)}{\partial t}\right)$

where $\gamma > 0$ and $\Phi(\cdot)$ is a matrix array of activation function $\phi(\cdot)$. Similarly, for the vector-valued time-varying problems [39], e.g., the system of linear equation $A(t)\mathbf{x}(t) = \mathbf{b}(t)$, with $A(t) \in \mathbb{R}^{(m\times n)}$, $\mathbf{x}(t) \in \mathbb{R}^n$, and $\mathbf{b}(t) \in \mathbb{R}^m$, the error function can be designed as $\mathbf{e}(t) = A(t)\mathbf{x}(t) - \mathbf{b}(t)$. Even for the scalar-valued time-varying problems [40–43], e.g., the time-varying 4th root finding problem $x^4(t) = a(t)$, with $a(t) \in \mathbb{R}$ and $x(t) \in \mathbb{R}$, the error function can be designed as $e(t) = x^4(t) - a(t)$. Note that the matrix-valued (or vector-valued) error function is a decoupled system and thus its ijth (or ith) subsystem, i.e., the scale-valued dynamical system $\dot{e}(t) = -\gamma\phi(e(t))$, can be used to analyze the corresponding convergence and stability. By exploiting ZNN design formula to solve different time-varying problems, various ZNN models that exploit the time-derivative information of coefficients can be constructed with their formulations shown in Table 1. In short, any ZNN model for solving any time-varying problem can be deemed as an equivalently expansion of the ZNN design formula.

Since Zhang et al. proposed ZNN models in the 2000s, modified models have been frequently proposed by considering different internal and external factors. Especially, when nonlinear activation functions are incorporated into the network models, stability research has gained significant progress. A brief review on the design of continuous-time ZNN models for various problems solving is presented in [47]. However, with the rapid development of the theory on ZNN, new variations have taken place [48] and in the ensuing part, we will briefly review some basic models of ZNN from different perspective.

2.1. Convergence and stability

In the research of neural networks, the key issues are convergence and stability. Broadly speaking, there are three ways for proving the convergence of ZNN models, i.e., proof based on Lyapunov theory, ordinary differential equation (ODE), or Laplace transform.

(1) *Proof based on Lyapunov theory [49]*. For example, for the time-varying nonlinear minimization problem solving with the task function being $f(\mathbf{x}(t), t) \in \mathbb{R}$ and $\mathbf{x}(t) \in \mathbb{R}^n$ in [36], the error function can be designed as

$$\mathbf{e}(t) = \frac{\partial f(\mathbf{x}(t), t)}{\partial \mathbf{x}(t)}.$$

By constructing a Lyapunov function candidate:

$$V(t) = \frac{1}{2}\mathbf{e}^{\mathrm{T}}(t)\mathbf{e}(t),$$

it can be concluded that $V(t)$ is evidently of the positive-definiteness. Then, computing its time derivative leads to

$$\dot{V}(t) = -\gamma\mathbf{e}^{\mathrm{T}}(t)\mathbf{e}(t),$$