Li et al. *Advances in Difference Equations* (2016) 2016:247
DOI 10.1186/s13662-016-0967-3

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Periodic orbits to Kaplan-Yorke like differential delay equations with two lags of ratio $(2k-1)/2$

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Abstract
In this paper, we study the periodic solutions to a type of differential delay equations with two lags of ratio $(2k-1)/2$ in the form

$$x'(t) = -f\big(x(t-2)\big) - f\big(x\big(t-(2k-1)\big)\big), \quad k \geq 2.$$

The $4k$-periodic solutions are obtained by using the variational method and the method of Kaplan-Yorke coupling system. This is a new type of differential-delay equations compared with all previous researches since the ratio of two lags is not an integer. Three functionals are constructed for a discussion on critical points. An example is given to demonstrate our main results.

MSC: 34B10; 34B15

Keywords: differential delay equation; periodic solutions; critical point theory; variational method

1 Introduction

The differential delay equations have useful applications in various fields such as age-structured population growth, control theory, and any models involving responses with nonzero delays [1–5].

Given $f \in C^0(R^+, R^-)$ with $f(-x) = -f(x), xf(x) > 0, x \neq 0$. Kaplan and Yorke [6] studied the existence of 4-periodic and 6-periodic solutions to the differential delay equations

$$x'(t) = -f\big(x(t-1)\big) \tag{1.1}$$

and

$$x'(t) = -f\big(x(t-1)\big) - f\big(x(t-2)\big), \tag{1.2}$$

respectively. The method they applied is transforming the two equations into adequate ordinary differential equations by regarding the retarded functions $x(t-1)$ and $x(t-2)$ as independent variables. They guessed that the existence of $2(n+1)$-periodic solution to the equation

$$x'(t) = -\sum_{i=1}^{n} f\bigl(x(t-i)\bigr) \tag{1.3}$$

could be studied under the restriction

$$x\bigl(t-(n+1)\bigr) = -x(t),$$

which was proved by Nussbaum [7] in 1978 by use of a fixed point theorem on cones.

After then, a lot of papers [8–21] discussed the existence and multiplicity of $2(n+1)$-periodic solutions to equation (1.3) and its extension

$$x'(t) = -\sum_{i=1}^{n} \operatorname{grad} F\bigl(x(t-i)\bigr), \tag{1.4}$$

where $F \in C^1(R^N, R), F(-x) = F(x), F(0) = 0$.

Recently, Zhang and Ge [22] studied the multiplicity of $2n$-periodic solutions to a type of differential delay equations of the form

$$x'(t) = -f\bigl(x(t-1)\bigr) - f\bigl(x(t-n)\bigr), \quad n \geq 2, \tag{1.5}$$

and obtained new results.

In this paper, we study the periodic orbits to a type of differential delay equations with two lags of ratio $(2k-1)/2$ in the form

$$x'(t) = -f\bigl(x(t-2)\bigr) - f\bigl(x\bigl(t-(2k-1)\bigr)\bigr), \quad k \geq 2, \tag{1.6}$$

which is different from (1.3) and can be regarded as a new extension of (1.2). The method applied in this paper is the variational approach in the critical point theory [23, 24].

Since the equation

$$x'(t) = -f\bigl(x(t-2)\bigr) - f\bigl(x(t-2k)\bigr), \quad k \geq 2,$$

can be changed into the form of equation (1.5) by the transformation

$$t = 2s, \qquad x(t) = x(2s) = y(s), \qquad \widehat{f}(y) = 2f(x),$$

this paper completes the research of the equations in the form

$$x'(t) = -f\bigl(x(t-2)\bigr) - f\bigl(x(t-n)\bigr), \quad n \geq 3. \tag{1.7}$$

In fact, it follows from

$$y'(s) = 2x'(t) = -2f\bigl(x(t-2)\bigr) - 2f\bigl(x(t-2k)\bigr) = -2f\bigl(x\bigl(2(s-1)\bigr)\bigr) - 2f\bigl(x\bigl(2(s-k)\bigr)\bigr)$$