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Fractional discrete-time of Hegselmann–Krause's type consensus model with numerical simulations

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ABSTRACT

The leader-following consensus problem of fractional-order multi-agent discrete-time systems is considered. In the system, interactions between opinions are defined like in the classical Hegselmann–Krause models but the memory is included by taking the fractional-order discrete-time operator on the left-hand side of the nonlinear system. In the paper we investigate various models for the dynamics of discrete-time fractional order opinions by analytical methods as well as by computer simulations.

1. Introduction

Speaking about consensus we need to imagine a group of individuals/experts who need to act together as a team or committee. Each of the experts has his own opinion but they should be willing to work together and to revise or exchange the opinion. The interactions between agents are usually described by interconnection topology of the system. In the literature, consensus denotes the agreement of a group faced with decision-making situations. Consensus has an old history, see for instance [1–5], whereas to our knowledge the models are linear and hence they are comparatively simple. This means in particular that the analysis needed can be carried out by linear techniques such as matrix theory, Markov chains and graph theory. As stated in [1] for a group behaviour the decision makers are more confident by sharing information with each other or consulting more than one expert. There has been an increasing interest in recent years in the analysis of multi-agent systems where agents interact according to some local rules. Each of the experts has his own opinion that could be changed by the influence of some interaction of its neighbours. The first nonlinear model was formulated and analysed in [6,7], where the generalisation of linear and inhomogeneous models into a state dependent one is studied. An extensive analysis of nonlinear models introduced by Krause in [7] or sometimes referred to as the Hegelsmann–Krause model was given in [8,9]. In papers [8,9] an extensive exploration of the nonlinear model with bounded confidence by a series of computer simulation is presented. In the past decades, there has been a great interest in distributed multi-agent coordination research due to their broad applications in the control community. The objective is to reach an agreement on information states, including positions, velocities, and attitudes, via local interaction. In our opinion there is other group of scientists who use controls to define the consensus for the systems. For most of the results the authors use continuous- or discrete-time dynamics with integer-order, see for instance [10–12]. In [10,11] the authors study the group consensus problems of heterogeneous multi-agent systems without and with time delays, respectively. The study of the leader-following consensus for multi-agent systems with nonlinear dynamics is presented in [12].

In nature many phenomena cannot be explained in the framework of integer-order dynamics, for example, the synchronised motion of agents in fractional circumstances, such as macromolecule fluids and porous media. In [13–15] the authors show that in order to demonstrate the stress–strain relationship one can use the fractional-order dynamics rather than integer-order dynamics. In addition, many other phenomena can naturally be explained by the coordinated behaviour of agents with fractional-order dynamics, see for instance [16–19], that are devoted for a class of fractional-order multi-agent systems. Moreover, fractional-order systems provide an

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excellent instrument for the description of memory. In the paper mentioned above there are considered particular observed consensus protocol with control, where in the definition of the consensus the controls are used. In our paper the right side of the model is like in the classical Hegselmann–Krause models, see [7–9], i.e. we study the nonlinear fractional order models with bounded confidence. We would like to stress that in our model we do not add any controls and in our case it is not possible to give the method of a construction of the controls in order to guarantee the consensus of the system like in [11,16–19]. In fact it is not easy to get the rigorous analytical results. For this reason we carry out the analysis of the presented nonlinear fractional order models to a large extent by computer simulations using "Maple". However we were able to prove some analytical results for some considered models. When we analyse the differences between opinions, we include an investigation of stability via $\mathcal{Z}$-transform of the system. Some additional results about stability of linear fractional systems can be found for example in [20]. However, in the paper we study consensus based on interactions between opinions taking into account memory inside them. In our opinion the memory is one of the most important factors that impacts on elements of groups behaviour. Since the definition of fractional operators take into account the "memory", i.e. the previous values of the function, we decided to include the memory by implementation of fractional operator instead of classical one.

In our paper, we come back to interactions between opinions defined like in Hegselmann–Krause models but with included memory by fractional-order operator on the left side. We use the Grünwald–Letnikov-type difference operator. As in that way our systems are positive, we start with positive initial opinions, then the trajectories are growing very fast. It decides about situation that to define a consensus by the classical way is no longer possible. Hence we decided to state new definition as tendency between trajectories, where those agents with the highest opinions inside the groups are called leaders. Such a type of consensus is called leader-following consensus. For the simplest case of just two agents it is not difficult to give a complete analysis of the dynamics. For an arbitrary n, however, the mathematical analysis of the fractional dynamics is rather difficult. For that reason we present in the paper an extensive analysis of this model for higher values of n agents by computer simulation.

2. Preliminaries

Let $c \in \mathbb{R}$ and $\mathbb{N}_c := \{c, c+1, c+2, \ldots\}$. Define the following sequence by its values:

$$a_k^{(\alpha)} := \begin{cases} 1 & \text{for } k = 0 \\ (-1)^k \dfrac{\alpha(\alpha-1)\ldots(\alpha-k+1)}{k!} & \text{for } k \in \mathbb{N}_1. \end{cases} \tag{1}$$

Since $\frac{\alpha(\alpha-1)\ldots(\alpha-k+1)}{k!} = \binom{\alpha}{k}$, the sequence $\left(a_k^{(\alpha)}\right)_{k\in\mathbb{N}_0}$ can be rewritten using the generalised binomial $\binom{\alpha}{k}$ as follows: $a_k^{(\alpha)} = (-1)^k \binom{\alpha}{k}$. Note that the generalised binomial $\binom{\alpha}{k}$ can be also calculated using the gamma function in the following way: $\binom{\alpha}{k} = \frac{\Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha-k+1)}$. However during computer calculations it is inadvisable to use the gamma function for computing the general binomial as the values of function Γ grow very fast. Therefore the best way is the possibility of using the recurrence formula for binomial coefficients. Note that $a_k^{(\alpha)}$ can be also defined in a recurrent way by

$$a_0^{(\alpha)} := 1,$$
$$a_{k+1}^{(\alpha)} := \left(1 - \frac{\alpha+1}{k+1}\right) a_k^{(\alpha)}, \quad k \in \mathbb{N}_0. \tag{2}$$

Relation (2) follows from the simple recalculations in the presentation with the above formula with function Γ. Then we have for instance $a_1^{(\alpha)} = -\alpha$ and $a_2^{(\alpha)} := \left(1 - \frac{\alpha+1}{2}\right) a_1^{(\alpha)} = \frac{\alpha(\alpha-1)}{2}$.

Proposition 1. *Let $\alpha \in (0, 1)$. Then the sequence $\left(a_k^{(\alpha)}\right)_{k\in\mathbb{N}_1}$ is increasing and $a_k^{(\alpha)} < 0$ for $k \in \mathbb{N}_1$.*

Proof. Firstly, let us notice that for $k \in \mathbb{N}_1$ we have $1 - \frac{\alpha+1}{k+1} > 0$, as $\alpha \in (0, 1)$. Then from (2) we get $a_{k+1}^{(\alpha)} < 0$ if only $a_k^{(\alpha)} < 0$ and as $a_1^{(\alpha)} = -\alpha < 0$ from the mathematical induction we have $a_k^{(\alpha)} < 0$ for all $k \in \mathbb{N}_1$. For the monotonicity let us calculate the following difference:

$$a_{k+1}^{(\alpha)} - a_k^{(\alpha)} = \left(1 - \frac{\alpha+1}{k+1}\right) a_k^{(\alpha)} - a_k^{(\alpha)} = -\frac{\alpha+1}{k+1} a_k^{(\alpha)}.$$

Hence if for all $k \in \mathbb{N}_1$ we have $a_k^{(\alpha)} < 0$, then the difference $a_{k+1}^{(\alpha)} - a_k^{(\alpha)}$ is positive and consequently, $a_{k+1}^{(\alpha)} > a_k^{(\alpha)}$. □

Now, let us state the properties of sequence $\left(a_k^{(-\alpha)}\right)_{k\in\mathbb{N}_0}$ with $\alpha \in [0, 1)$.

Proposition 2. *Let $\alpha \in (0, 1)$. Then the sequence $\left(a_k^{(-\alpha)}\right)_{k\in\mathbb{N}_1}$ is decreasing and $a_k^{(-\alpha)} > 0$ for $k \in \mathbb{N}_1$.*

Proof. Firstly, let us notice that $a_0^{(-\alpha)} = 1$ and $a_k^{(-\alpha)} = \frac{\alpha(\alpha+1)\cdot\ldots\cdot(\alpha+k-1)}{k!}$ for $k \in \mathbb{N}_1$. Similarly, as in the proof of Proposition 1 one can show that for $\alpha \in [0, 1)$ we have $0 < 1 - \frac{1}{k+1} \le 1 - \frac{-\alpha+1}{k+1} < 1$ for $k \ge 1$. Then from (2) we get $a_{k+1}^{(-\alpha)} > 0$ if only $a_k^{(-\alpha)} > 0$. Additionally, as $a_1^{(-\alpha)} = \alpha > 0$ from the mathematical induction $a_k^{(\alpha)} > 0$ for all $k \in \mathbb{N}_1$. Observe that

$$a_{k+1}^{(-\alpha)} - a_k^{(-\alpha)} = \left(1 - \frac{-\alpha+1}{k+1}\right) a_k^{(-\alpha)} - a_k^{(-\alpha)} = -\frac{-\alpha+1}{k+1} a_k^{(-\alpha)}.$$

Hence if for all $k \in \mathbb{N}_1$ holds that $a_k^{(-\alpha)} > 0$, then the difference $a_{k+1}^{(-\alpha)} - a_k^{(-\alpha)}$ is negative for all $k \ge 1$ and consequently, $a_{k+1}^{(-\alpha)} < a_k^{(-\alpha)}$. □

Using Proposition 2 we can easily deduce that sequence $\left(a_k^{(-\alpha)}\right)_{k\in\mathbb{N}_q}$ is convergent.

Corollary 3. *If $\alpha \in [0, 1)$, then $\lim_{k\to\infty} a_k^{(-\alpha)} = 0$.*

Proof. For $\alpha = 0$ we get $a_0^{(0)} = 1$ and $a_k^{(0)} = 0$ for $k \ge 1$. Now let us take $\alpha \in (0, 1)$. Then using property of gamma function we get

$$a_k^{(-\alpha)} = (-1)^k \binom{-\alpha}{k} = \binom{k+\alpha-1}{k} = \frac{\Gamma(k+\alpha)}{\Gamma(k+1)\Gamma(\alpha)} = \frac{1 + O\left(k^{-1}\right)}{k^{1-\alpha}\Gamma(\alpha)}.$$

Hence since $\alpha \in (0, 1)$, we get $\Gamma(\alpha) > 0$, $0 < 1 - \alpha < 1$ and consequently, $\lim_{k\to\infty} a_k^{(-\alpha)} = 0$. □

Let us recall that the $\mathcal{Z}$-transform of a sequence $\left(y(n)\right)_{n\in\mathbb{N}_0}$ is a complex function given by

$$Y(z) := \mathcal{Z}[y](z) = \sum_{k=0}^{\infty} \frac{y(k)}{z^k},$$

where $z \in \mathbb{C}$ denotes a complex number for which this series converges absolutely. The $\mathcal{Z}$–transform can be extended to vector valued sequences in the componentwise manner. Note that since

$$a_k^{(\alpha)} = (-1)^k \binom{\alpha}{k} = \binom{k-\alpha-1}{k},$$

then for $|z| > 1$ and $\alpha \in \mathbb{R}$ we have

$$\mathcal{Z}\left[a_k^{(\alpha)}\right](z) = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} z^{-k} = \left(1 - \frac{1}{z}\right)^{\alpha}. \tag{3}$$

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