Neurocomputing ∎ (∎∎∎∎) ∎∎∎–∎∎∎

Contents lists available at ScienceDirect

Neurocomputing

journal homepage: www.elsevier.com/locate/neucom

Brief papers

# A combinatorial necessary and sufficient condition for cluster consensus

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ARTICLE INFO

*Article history:*
Received 10 November 2015
Received in revised form
12 May 2016
Accepted 7 August 2016
Communicated by H. Zhang

*Keywords:*
Cluster consensus
Multi-agent system
Linear switched System
Cooperative control

ABSTRACT

In this letter, cluster consensus of discrete-time linear multi-agent systems is investigated. A set of stochastic matrices $\mathcal{P}$ is said to be a cluster consensus set if the system achieves cluster consensus for any initial state and any sequence of matrices taken from $\mathcal{P}$. By introducing a cluster ergodicity coefficient, we present an equivalence relation between a range of characterization of cluster consensus set under some mild conditions including the widely adopted inter-cluster common influence. We obtain a combinatorial necessary and sufficient condition for a compact set $\mathcal{P}$ to be a cluster consensus set. This combinatorial condition is an extension of the avoiding set condition for global consensus, and can be easily checked by an elementary routine. As a byproduct, our result unveils that the cluster-spanning tree condition is not only sufficient but necessary in some sense for cluster consensus problems.



## 1. Introduction

In the past two decades, consensus problems in multi-agent systems have gained increasing attention in various research communities, ranging from formation of unmanned air vehicles to data fusion of sensor networks, from swarming of animals to synchronization of distributed oscillators [1–4]. The main objective of consensus problems is to design appropriate protocols and algorithms such that the states of a group of agents converge to a consistent value (see [5,6] for a survey of this prolific field). In many distributed consensus algorithms, the agents update their values as linear combinations of the values of agents with which they can communicate:

$$x_i(t+1) = \sum_j p_{ij}(t+1)x_j(t), \tag{1}$$

where $x_i(t)$ is the value of agent $i$ and $P(t) = (p_{ij}(t))$ for every discrete time instant $t \ge 0$ represents a *stochastic* matrix, i.e., $p_{ij}(t) \ge 0$ and $\sum_j p_{ij}(t) = 1$. The states of agents following such linear averaging algorithms tend to get closer over time. The problem of characterizing the complete sequence of matrices $P(t)$ for consensus is however known to be notoriously difficult [7]. A moderate goal would be to determine whether the system (1) converges to a state of consensus for all sequences of matrices $P(t)$ in a certain set $\mathcal{P}$. Remarkably, Blondel and Olshevsky in a recent work [8] presented an explicit combinatorial condition, which is both necessary and sufficient for consensus of (1) in that sense. This condition, dubbed "avoiding sets condition", is easy to check with an algorithm and thus the consensus problem is decidable.

While most existing works are concerned with global consensus (namely, all the agents reach a common state), in various real-world applications, there may be multiple consistent states as agents in a network often split into several groups to carry out different cooperative tasks. Typical situations include obstacle avoidance of animal herds, team hunting of predators, social learning under different environments, coordinated military operations, and task allocation over the network between groups. A possible solution is given by the cluster (or group) consensus algorithms [9,10], where the agents in a network are divided into multiple subnetworks and different subnetworks can reach different consistent states asymptotically. Evidently, cluster consensus is an extension of (global) consensus. Various sufficient conditions and necessary conditions (although much fewer) for cluster consensus have been reported in the literature for discrete-time systems [10–14], simple first- or second-order continuous-time systems [9,15–21], and high-order dynamics [22], to name a few. However, most of these conditions rely on either complicated linear matrix inequalities or algebraic conditions involving eigenvalues of the system matrices, which are in general difficult to check.

With the above inspiration, we aim to work on efficiently verifiable conditions for cluster consensus by extending the results in [8] for global consensus, which are highly non-trivial. The main

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http://dx.doi.org/10.1016/j.neucom.2016.08.025
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contribution of this paper is to establish a combinatorial necessary and sufficient condition which guarantees the cluster consensus of system (1) under some common assumptions, i.e., self-loops, either undirected graph or doubly stochastic state-update matrices, and inter-cluster common influence. Some of the previous convergence criteria can be quickly reproduced from our results. It is noteworthy that the authors in [10] showed that, under some mild assumptions, the cluster consensus of (1) can be achieved provided the graphs associated with $P(t)$ contain cluster-spanning trees. Interestingly, our result implies that the cluster-spanning trees condition is essentially necessary.

## 2. Preliminaries

In this section, some definitions and lemmas on graph theory and matrix theory are given as the preliminaries. We refer the reader to the textbooks [6,23] for more details.

Let $G=(V,E)$ be a directed graph of order $n$ with the set of vertices $V=\{1,\ldots,n\}$ and the set of edges $E\subseteq V\times V$. For a stochastic matrix $P=(p_{ij})\in\mathbb{R}^{n\times n}$ (namely, $p_{ij}\ge 0$ and $\sum_{j=1}^n p_{ij}=1$ for all $i,j$), a corresponding directed graph $G(P)=(V,E)$ can be constructed by taking $V=\{1,\ldots,n\}$, and $E=\{(i,j): p_{ij}>0\}$. $G(P)$ is assumed to be unweighted throughout the paper. Given a subset $S\subseteq V$, denote by $N_G(S)$ the set of out-neighbors of $S$ in $G$, i.e., $N_G(S)=\{j\in V: i\in S, (i,j)\in E\}$. A directed path from vertex $i$ to $j$ of length $l$ is a sequence of edges $(i,i_1),(i_1,i_2),\ldots,(i_l,j)$ with distinct vertices $i_1,\ldots,i_l\in V$. If $(i,i)\in E$, then there exists a self-loop at vertex $i$.

A *clustering* $\mathcal{C}=\{C_1,\ldots,C_K\}$ of the directed graph $G$ is defined by dividing its vertex set into disjoint clusters $\{C_k\}_{k=1}^K$. In other words, $\mathcal{C}$ satisfies $\cup_{k=1}^K C_k=V$ and $C_k\cap C_{k'}=\varnothing$ for $k\ne k'$. Letting $x(t)=(x_1(t),\ldots,x_n(t))^T$, we recast the system (1) as

$$x(t+1)=P(t+1)x(t). \tag{2}$$

**Definition 1.** For a given clustering $\mathcal{C}=\{C_1,\ldots,C_K\}$, a set of $n\times n$ stochastic matrices $\mathcal{P}$ is said to be a *cluster consensus set* if for any initial state $x(0)$ and all sequences $P(1),P(2),\ldots,\in\mathcal{P}$,

$$\lim_{t\to\infty} x(t)=\sum_{k=1}^K \alpha_k \mathbf{1}_{C_k}, \tag{3}$$

where $\mathbf{1}_{C_k}$ is the sum of $i$th $n$-dimensional basis vector $e_i=(0,\ldots,0,\overset{i\text{th}}{1},0,\ldots,0)^T$ over all $i\in C_k$, and $\alpha_k$ is some scalar.

**Remark 1.** In most of the literature, given a sequence of stochastic matrices (or switching signal) $P(1),P(2),\ldots$, the system (2) is said to achieve *cluster consensus* if (3) holds for any initial state $x(0)$. This is also referred to as *intra-cluster synchronization* in [10,18,24], where the cluster consensus requires additionally the separation of states of agents in different clusters. Nevertheless, the inter-cluster separation can only be realized by incorporating adapted external inputs.

**Definition 2** (*[10]*). A stochastic matrix $P$ is said to have *inter-cluster common influence* if for all $k\ne k'$, $\sum_{j\in C_{k'}} p_{ij}$ is identical with respect to all $i\in C_k$.

**Remark 2.** Since the entries on each row of $P$ sum up to one, the above statement automatically holds for $k=k'$ if $P$ has inter-cluster common influence. Therefore, $\sum_{j\in C_{k'}} p_{ij}$ depends only on $k$ and $k'$. This (and some closely related variants) is a common assumption for cluster consensus problems; see e.g. [9–13,15,24,25]. It is direct to check that if $P_1$ and $P_2$ have inter-cluster common influence with respect to the same clustering $\mathcal{C}$, so does $P_1P_2$.

To analyze the cluster consensus of the multi-agent system (2), we will need to estimate some characteristics of infinite product of stochastic matrices. For a stochastic matrix $P=(p_{ij})\in\mathbb{R}^{n\times n}$, we define the cluster ergodicity coefficient with respect to a clustering $\mathcal{C}$ as

$$\tau_{\mathcal{C}}(P)=\frac{1}{2}\max_{1\le k\le K}\max_{i,j\in C_k}\sum_{s=1}^n |p_{is}-p_{js}|=\frac{1}{2}\max_{1\le k\le K}\max_{i,j\in C_k}\| p_i-p_j\|_1, \tag{4}$$

where $p_i=(p_{i1},\ldots,p_{in})$ is the $i$th row of $P$ and $\|\cdot\|_1$ represents the 1-norm of vector.

It can be seen that $0\le\tau_{\mathcal{C}}(P)\le 1$ and that $\tau_{\mathcal{C}}(P)=0$ if and only if $P=\sum_{k=1}^K \mathbf{1}_{C_k}y_k^T$, where $y_k$ is a stochastic vector, namely, $P$ has identical rows for each cluster. Hence, $\tau_{\mathcal{C}}$ can be viewed as an extension of the well-known Dobrushin ergodicity coefficient [23] for clustering.

**Lemma 1.** *If $P_1=(p'_{ij})$ and $P_2=(p''_{ij})$ are two $n\times n$ stochastic matrices having inter-cluster common influence with respect to the same clustering $\mathcal{C}$, then*

$$\tau_{\mathcal{C}}(P_1P_2)\le\tau_{\mathcal{C}}(P_1)\tau_{\mathcal{C}}(P_2)\le\min\{\tau_{\mathcal{C}}(P_1),\tau_{\mathcal{C}}(P_2)\}.$$

**Proof.** We only need to show the first inequality. Suppose that $\mathcal{C}=\{C_1,\ldots,C_K\}$. We first recall a useful lemma (see [26, p. 126, Lemma 1.1]): For any stochastic matrix $P=(p_{ij})\in\mathbb{R}^{n\times n}$ and $i,j\in V=\{1,\ldots,n\}$,

$$\frac{1}{2}\sum_{s=1}^n |p_{is}-p_{js}|=\max_{A\subseteq V}\sum_{s\in A}(p_{is}-p_{js}). \tag{5}$$

It follows immediately from (5) that

$$\tau_{\mathcal{C}}(P_1P_2)=\max_{1\le k\le K}\max_{i,j\in C_k}\max_{A\subseteq V}\sum_{s\in A}\sum_{l=1}^n (p'_{il}-p'_{jl})p''_{ls}.$$

Denote by $f^+=\max\{f,0\}$ and $f^-=-\min\{f,0\}$ for $f\in\mathbb{R}$. Hence, $f=f^+-f^-$ and $|f|=f^++f^-$. Fix $1\le k\le K$ and $i,j\in C_k$. For any $1\le k'\le K$, we have $0=\sum_{l\in C_{k'}}(p'_{il}-p'_{jl})=\sum_{l\in C_{k'}}(p'_{il}-p'_{jl})^+-\sum_{l\in C_{k'}}(p'_{il}-p'_{jl})^-$ since $P_1$ has inter-cluster common influence. Accordingly,

$$\sum_{l\in C_{k'}}(p'_{il}-p'_{jl})^+=\sum_{l\in C_{k'}}(p'_{il}-p'_{jl})^-=\frac{1}{2}\sum_{l\in C_{k'}}|p'_{il}-p'_{jl}|.$$

In view of this relation, we obtain

$$\begin{aligned}
\sum_{s\in A}\sum_{l=1}^n (p'_{il}-p'_{jl})p''_{ls} &= \sum_{s\in A}\sum_{1\le k'\le K}\sum_{l\in C_{k'}}(p'_{il}-p'_{jl})p''_{ls}\\
&= \sum_{1\le k'\le K}\sum_{l\in C_{k'}}(p'_{il}-p'_{jl})^+\sum_{s\in A}p''_{ls}\\
&\quad -\sum_{1\le k'\le K}\sum_{l\in C_{k'}}(p'_{il}-p'_{jl})^-\sum_{s\in A}p''_{ls}\\
&\le \sum_{1\le k'\le K}\left(\frac{1}{2}\sum_{l\in C_{k'}}|p'_{il}-p'_{jl}|\right)\max_{l\in C_{k'}}\sum_{s\in A}p''_{ls}\\
&\quad -\sum_{1\le k'\le K}\left(\frac{1}{2}\sum_{l\in C_{k'}}|p'_{il}-p'_{jl}|\right)\min_{l\in C_{k'}}\sum_{s\in A}p''_{ls}\\
&\le \sum_{1\le k'\le K}\left(\frac{1}{2}\sum_{l\in C_{k'}}|p'_{il}-p'_{jl}|\right)\max_{l,l'\in C_{k'}}\\
&\quad \sum_{s\in A}(p''_{ls}-p''_{l's})\\
&\le \left(\frac{1}{2}\sum_{l=1}^n|p'_{il}-p'_{jl}|\right)\cdot\max_{1\le k'\le K}\max_{l,l'\in C_{k'}}\sum_{s\in A}(p''_{ls}-p''_{l's}).
\end{aligned}$$

Please cite this article as: Y. Shang, A combinatorial necessary and sufficient condition for cluster consensus, Neurocomputing (2016), http://dx.doi.org/10.1016/j.neucom.2016.08.025