Computational Geometry 64 (2017) 1–12

Contents lists available at ScienceDirect

Computational Geometry: Theory and Applications

www.elsevier.com/locate/comgeo

Common developments of three incongruent boxes of area 30 ☆

Dawei Xu [a,*,1], Takashi Horiyama [b,2], Toshihiro Shirakawa [c], Ryuhei Uehara [a,3]

[a] School of Information Science, Japan Advanced Institute of Science and Technology, Nomi, Ishikawa 923-1292, Japan
[b] Information Technology Center, Saitama University, Saitama, Japan
[c] Amateur Mathematician, Japan

ARTICLE INFO

Article history:
Received 15 March 2016
Accepted 13 March 2017
Available online 18 March 2017

Keywords:
Common development
Convex polyhedron
Zero-suppressed binary decision diagrams

ABSTRACT

We investigate common developments that can fold into plural incongruent orthogonal boxes. Recently, it was shown that there are infinitely many orthogonal polygons that fold into three boxes of different size. However, the smallest one that folds into three boxes consists of 532 unit squares. From the necessary condition, the smallest possible surface area that can fold into two boxes is 22, and the smallest possible surface area for three different boxes is 46. For the area 22, it has been shown that there are 2,263 common developments of two boxes by exhaustive search. However, the area 46 is too huge to search. In this paper, we focus on the polygons of area 30, which is the second smallest area of two boxes that admits to fold into two boxes of size $1 \times 1 \times 7$ and $1 \times 3 \times 3$. Moreover, when we fold along diagonal lines of rectangles of size 1×2, this area 30 may admit to fold into a box of size $\sqrt{5} \times \sqrt{5} \times \sqrt{5}$. The results are summarized as follows. There exist 1,080 common developments of two boxes of size $1 \times 1 \times 7$ and $1 \times 3 \times 3$. Among them, there are nine common developments of three boxes of size $1 \times 1 \times 7$, $1 \times 3 \times 3$, and $\sqrt{5} \times \sqrt{5} \times \sqrt{5}$. Interestingly, one of nine such polygons folds into three different boxes $1 \times 1 \times 7$, $1 \times 3 \times 3$, and $\sqrt{5} \times \sqrt{5} \times \sqrt{5}$ in four different ways.

1. Introduction

Since Lubiw and O'Rourke posed the problem in 1996 [11], polygons that can fold into a (convex) polyhedron have been investigated in the area of computational geometry. In general, we can state the development/folding problem as follows:

Input : A polygon P and a polyhedra Q
Output: Determine whether P can fold into Q or not

☆ A preliminary version was presented at TAMC 2015, Singapore.

* Corresponding author.
E-mail addresses: xudawei@jaist.ac.jp (D. Xu), horiyama@al.ics.saitama-u.ac.jp (T. Horiyama), 70ku51n@gmail.com (T. Shirakawa), uehara@jaist.ac.jp (R. Uehara).

[1] Dawei Xu is supported in part by CSC Grant Number 201508050011.
[2] Takashi Horiyama is supported in part by JSPS KAKENHI Grant Number 15K00008 and MEXT KAKENHI Grant Number 24106007.
[3] Ryuhei Uehara is supported in part by JSPS KAKENHI Grant Number 26330009 and MEXT KAKENHI Grant Number 24106004.

http://dx.doi.org/10.1016/j.comgeo.2017.03.001
0925-7721/

Fig. 1. Cubigami.

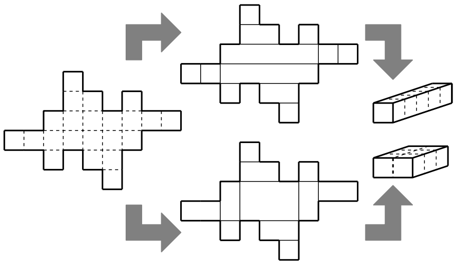

Fig. 2. A polygon folding into two boxes of size $1 \times 1 \times 5$ and $1 \times 2 \times 3$ in [13].

When Q is a tetramonohedron (a tetrahedron with four congruent triangular faces), Akiyama and Nara gave a complete characterization of P by using the notion of tiling [3,4]. Except that, we have quite a few results from the mathematical viewpoint. Hence, we can tackle this problem from the viewpoint of computational geometry and algorithms. For example, Demaine and O'Rourke gave an $O(n^3)$ time algorithm for a convex polyhedron Q in [7]. Precisely, the algorithm determines if P can fold into *some convex* polyhedron Q, however, it does not give the concrete *shape* of Q. The algorithm computes the matching of edges for gluing in P to fold into a convex polyhedron Q, and it also determines the vertices of Q by checking the curvature of the vertices. However, it does not give the crease pattern of P for folding, which forms the set of edges on Q. We remark that the shape is uniquely determined from its convexity of Q by the Alexandrov's Theorem (see [9]). Moreover, such a shape can be determined by the pseudopolynomial time algorithm for Alexandrov's Theorem proposed in [9]. However, the algorithm in [9] runs in $O(n^{456.5})$ time (with some geometric parameters), and hence it is not practical so far. Thus, we still have to decide the crease lines to make the shape by experiment. In other words, it is quite difficult to determine the *shape* Q obtained from a polygon P by folding even if we can find its edge matching of gluing. Recently, some restricted cases were discussed in [2]; they give efficient algorithms for the cases that P is a petal polygon, and Q is a pyramid or its variants.

From the viewpoint of computation, one natural restriction is that considering the orthogonal polygons and polyhedra which consist of unit squares and unit cubes, respectively. Such polygons have many applications including toys and puzzles. For example, the puzzle "cubigami" (Fig. 1) is a common development of all tetracubes except one (since the last one has surface area 16, while the others have surface area 18), which is developed by Miller and Knuth. Some related results can be found in the books on geometric folding algorithms by Demaine and O'Rourke [7,15]. One of the many interesting problems in this area asks whether there exists a polygon that folds into plural incongruent orthogonal boxes. This folding problem is very natural but quite counterintuitive; for a given polygon that consists of unit squares, the problem asks are there two or more ways to fold it into simple convex orthogonal polyhedra (Fig. 2). Biedl et al. first gave two polygons that fold into two incongruent orthogonal boxes [6] (see also Figure 25.53 in the book by Demaine and O'Rourke [7]). Later, Mitani and Uehara constructed infinite families of orthogonal polygons that fold into two incongruent orthogonal boxes [13]. Recently, Shirakawa and Uehara extended the result to three boxes in a nontrivial way; they showed infinite families of orthogonal polygons that fold into three incongruent orthogonal boxes [17]. However, the smallest polygon by their method contains 532 unit squares, and it is still open if there exists much smaller polygon of several dozens of squares that folds into three (or more) different boxes.

It is easy to see that two boxes of size $a \times b \times c$ and $a' \times b' \times c'$ can have a common development only if they have the same surface area, i.e., when $2(ab + bc + ca) = 2(a'b' + b'c' + c'a')$ holds. We can compute small surface areas that may admit to fold into two or more boxes by a simple exhaustive search. We show a part of the table for $1 \le a \le b \le c \le 50$ in Table 1. From the table, we can say that the smallest surface area is at least 22 to have a common development of two boxes, and then their sizes are $1 \times 1 \times 5$ and $1 \times 2 \times 3$. In fact, Abel et al. have confirmed that there exist 2,263 common developments of two boxes of size $1 \times 1 \times 5$ and $1 \times 2 \times 3$ by an exhaustive search [1]. On the other hand, the smallest surface area that may admit to fold into three boxes is 46, which may fold into three boxes of size $1 \times 1 \times 11$, $1 \times 2 \times 7$, and $1 \times 3 \times 5$. However, the number of polygons of area 46 seems to be too huge to search. This number is strongly related