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Reversibility and foldability of Conway tiles

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In this paper, we proved that an arbitrary Conway tile is reversible to another Conway tile. We also determine all reversible pairs of figures, both of which tile the plane. Then we prove that the set of all nets of an isotetrahedron is closed under some reversible operation. Finally, we prove that a regular Conway tile is foldable into an isotetrahedron.

1. Definitions and known results

Haberdasher's puzzle asks one to dissect an equilateral triangle (referred to as P) into several pieces and rearrange them to make a square (referred to as Q) after hinging the pieces like a chain (Fig. 1.1).

Scrutinizing the essence of the Haberdasher's puzzle, a reversible transformation between a pair of figures P and Q is defined in [5] as follows.

A pair of figures P and Q is said to be **reversible** (or **equi-rotational**) if P and Q satisfy the following conditions:

1. There exists a dissection tree DT along which P is dissected into n pieces.
2. Hinge n pieces at $n-1$ endvertices of DT to make a chain of n pieces.
3. Fix an endpiece of the chain of pieces and rotate the remaining pieces clockwise, counter-clockwise to obtain P, Q respectively.
4. During the rotation, motions of all pieces around every hinge are either all clockwise or counterclockwise.
5. In this reversible transformation, all dissection lines of P (i.e., edges of DT) are located on the perimeter of Q and all perimeter parts of P are located in the interior of Q, and vice-versa (**reversible condition**).

Abbott et al. proved in [1] that every pair of polygons P and Q with the same area is hinge transformable if we do not require the above conditions 4 and 5. Under these conditions, hinge transformable figures have some remarkable properties which we will discuss in this article. Many other studies on this topic are found in [7–9]. We consider a chain consisting of $n+1$ vertices $v_0, v_1, v_2, \dots, v_n$ and n edges (straight or curved segments) $e_1 = v_0v_1, e_2 = v_1v_2, \dots, e_n = v_{n-1}v_n$. We denote this chain by P_n (Fig. 1.2(b)).

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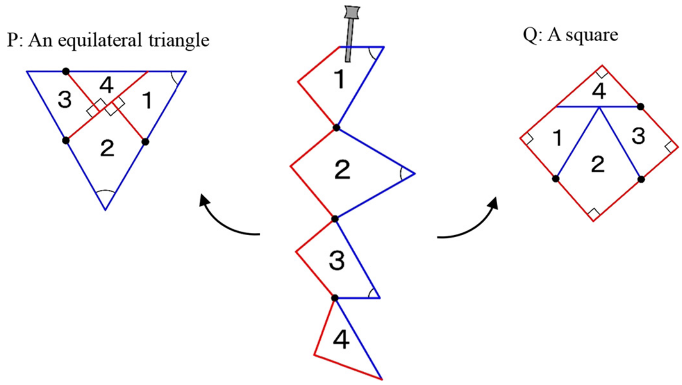

Fig. 1.1. Illustration of Haberdasher's puzzle.

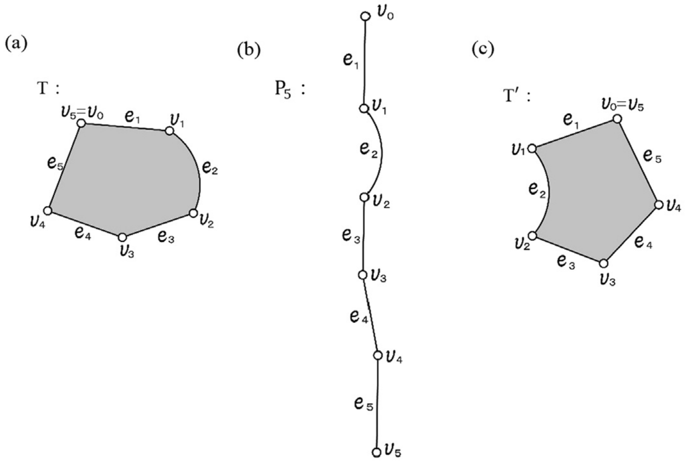

Fig. 1.2. A trunk T and a conjugate trunk T'.

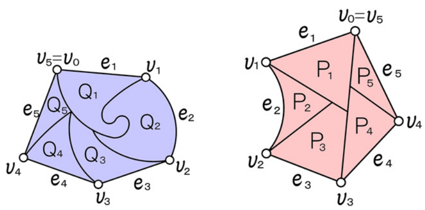

Fig. 1.3. Dissected T and dissected T' along D_1, D_2, respectively.

By connecting v_0 with v_n of P_n, a cycle C_n is obtained. Any planar region (or topological disk) surrounded by C_n is called a **trunk** or a **conjugate trunk** when the edges of C_n appear $e_1, e_2, e_3, \ldots, e_n$ or $e_n, e_{n-1}, \ldots, e_2, e_1$ in clockwise direction, respectively. A trunk and a conjugate trunk are denoted by T, T', respectively (Fig. 1.2(a), (c)).

Notice that there are infinitely many conjugate trunks T' for a given trunk T if $n \geq 4$. Let D_1, D_2 be dissection trees spanning all vertices of C_n and dissect T, T' along D_1, D_2, respectively (Fig. 1.3).

Unhinge the vertex $v_n = v_0$ of T, T', respectively. Then two chains (see Fig. 1.4(a), (c)), where one has the pieces Q_i $(i = 1, 2, \ldots, n)$ of T on the left side of the chain and the other has the pieces P_i of T' on the right side of the chain, are obtained.

Combine the chains derived from T and T' such that each e_i has a piece P_i on the right side and a piece Q_i on the left side. The chain obtained in this manner is called a **double chain of** (T, T') (or simply a **double chain**) (as shown in Fig. 1.4(b)).

A piece of a double chain is **empty** if at least one-side of the piece consists of only a perimeter part e_i (see the right side of piece **4** of the chain in Fig. 1.1). If a double chain has an empty piece, then we distinguish one side of that edge from the other side so that it satisfies the reversible condition of reversible transformation. If one of the endpieces (say P_1 and Q_1 in Fig. 1.5) of the double chain of (T, T') is fixed and the remaining pieces are rotated clockwise or counterclockwise, then