Computational Geometry 63 (2017) 1–9

Contents lists available at ScienceDirect

Computational Geometry: Theory and Applications

www.elsevier.com/locate/comgeo

An optimal algorithm for plane matchings in multipartite geometric graphs ☆

Ahmad Biniaz [a,*], Anil Maheshwari [a], Subhas C. Nandy [b], Michiel Smid [a]

[a] Carleton University, Ottawa, Canada
[b] Indian Statistical Institute, Kolkata, India

A R T I C L E I N F O

Article history:
Received 7 July 2016
Accepted 15 February 2017
Available online 24 February 2017

Keywords:
Colored point sets
Balanced cut
Plane matchings

A B S T R A C T

Let P be a set of n points in general position in the plane which is partitioned into *color* classes. The set P is said to be *color-balanced* if the number of points of each color is at most $\lfloor n/2 \rfloor$. Given a color-balanced point set P, a *balanced cut* is a line which partitions P into two color-balanced point sets, each of size at most $2n/3+1$. A *colored matching* of P is a perfect matching in which every edge connects two points of distinct colors by a straight line segment. A *plane colored matching* is a colored matching which is non-crossing. In this paper, we present an algorithm which computes a balanced cut for P in linear time. Consequently, we present an algorithm which computes a plane colored matching of P optimally in $\Theta(n \log n)$ time.

1. Introduction

Let P be a set of n points in general position (no three points on a line) in the plane. Assume P is partitioned into *color* classes, i.e., each point in P is colored by one of the given colors. P is said to be *color-balanced* if the number of points of each color is at most $\lfloor n/2 \rfloor$. In other words, P is color-balanced if no color is in strict majority. For a color-balanced point set P, we define a *feasible cut* as a line ℓ which partitions P into two point sets Q_1 and Q_2 such that both Q_1 and Q_2 are color-balanced. In addition, if the number of points in each of Q_1 and Q_2 is at most $2n/3+1$, then ℓ is said to be a *balanced cut*. We note that a feasible cut may passes through one or two points of P. The well-known ham-sandwich cut (see [11]) is a balanced cut: given a set of $2m$ red points and $2m$ blue points in general position in the plane, a ham-sandwich cut is a line ℓ which partitions the point set into two sets, each of them having m red points and m blue points. Feasible cuts and balanced cuts are useful for convex partitioning of the plane and for computing plane structures, e.g., plane matchings and plane spanning trees.

Assume n is an even number. Let $\{R, B\}$ be a partition of P such that $|R| = |B| = n/2$. Let $K_n(R, B)$ be the complete bipartite geometric graph on P which connects every point in R to every point in B by a straight-line edge. An *RB-matching* in P is a perfect matching in $K_n(R, B)$. Assume the points in R are colored red and the points in B are colored blue. An RB-matching in P is also referred to as a *red-blue matching* or a *bichromatic matching*. A *plane RB-matching* is an RB-matching in which no two edges cross. Let $\{P_1, \dots, P_k\}$, where $k \geq 2$, be a partition of P. Let $K_n(P_1, \dots, P_k)$ be the

☆ Research supported by NSERC. A preliminary version of this paper appeared in Proceedings of the 14th Algorithms and Data Structures Symposium, WADS 2015.

* Corresponding author.
E-mail address: ahmad.biniaz@gmail.com (A. Biniaz).

http://dx.doi.org/10.1016/j.comgeo.2017.02.004
0925-7721/

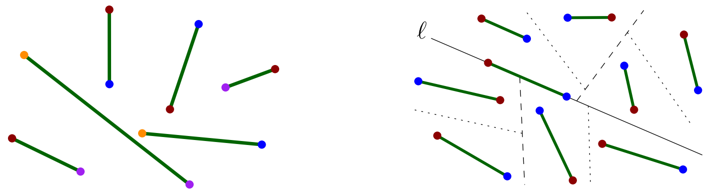

Fig. 1. (a) A plane colored matching. (b) Recursive ham sandwich cuts. (For interpretation of the references to color in this figure, the reader is referred to the web version of this article.)

complete multipartite geometric graph on P which connects every point in P_i to every point in P_j by a straight-line edge, for all $1 \le i < j \le k$. Imagine the points in P to be colored, such that all the points in P_i have the same color, and for $i \ne j$, the points in P_i have a different color from the points in P_j. We say that P is a *k-colored* point set. A *colored matching* of P is a perfect matching in $K_n(P_1, \ldots, P_k)$. A *plane colored matching* of P is a perfect matching in $K_n(P_1, \ldots, P_k)$ in which no two edges cross. See Fig. 1(a).

In this paper we consider the problem of computing a balanced cut for a given color-balanced point set in general position in the plane. We show how to use balanced cuts to compute plane matchings in multipartite geometric graphs.

1.1. Previous work on 2-colored point sets

Let P be a set of $n = 2m$ points in general position in the plane. Let $\{R, B\}$ be a partition of P such that $|R| = |B| = m$. Assume the points in R are colored red and the points in B are colored blue. It is well-known that $K_n(R, B)$ has a plane RB-matching [1]. In fact, a minimum weight RB-matching, i.e., a perfect matching that minimizes the total Euclidean length of the edges, is plane. A minimum weight RB-matching in $K_n(R, B)$ can be computed in $O(n^{2.5} \log n)$ time [15], or even in $O(n^{2+\epsilon})$ time [2]. Consequently, a plane RB-matching can be computed in $O(n^{2+\epsilon})$ time. As a plane RB-matching is not necessarily a minimum weight RB-matching, one may compute a plane RB-matching faster than computing a minimum weight RB-matching. Hershberger and Suri [8] presented an $O(n \log n)$ time algorithm for computing a plane RB-matching. They also proved a lower bound of $\Omega(n \log n)$ time for computing a plane RB-matching, by providing a reduction from sorting.

Alternatively, one can compute a plane RB-matching by recursively applying the ham sandwich theorem; see Fig. 1(b). We say that a line ℓ *bisects* a point set R if both sides of ℓ have the same number of points of R. If $|R|$ is odd, the line ℓ contains one point of R, and if $|R|$ is even, the line ℓ may contain zero or two points of R.

Theorem 1 *(Ham sandwich theorem; see [11]). For a point set P in general position in the plane which is partitioned into sets R and B, there exists a line that simultaneously bisects R and B.*

A line ℓ that simultaneously bisects R and B can be computed in $O(|R| + |B|)$ time, assuming $R \cup B$ is in general position in the plane [11]. By recursively applying Theorem 1, we can compute a plane RB-matching in $\Theta(n \log n)$ time.

1.2. Previous work on 3-colored point sets

Let P be a set of $n = 3m$ points in general position in the plane. Let $\{R, G, B\}$ be a partition of P such that $|R| = |G| = |B| = m$. Assume the points in R are colored red, the points in G are colored green, and the points in B are colored blue. A lot of research has been done to generalize the ham sandwich theorem to 3-colored point sets, see e.g. [4,5,10]. It is easy to see that there exist configurations of P such that there exists no line which bisects R, G, and B, simultaneously. Furthermore, for some point sets P, for any $k \in \{1, \ldots, m-1\}$, there does not exist any line ℓ such that an open half-plane bounded by ℓ contains k red, k green, and k blue points (see [5] for an example). For the special case, where the points on the convex hull of P are monochromatic, Bereg and Kano [5] proved that there exists an integer $1 \le k \le m-1$ and an open half-plane containing exactly k points from each color.

Bereg et al. [4] proved that if the points of P are on any closed Jordan curve γ, then for every integer k with $0 \le k \le m$ there exists a pair of disjoint intervals on γ whose union contains exactly k points of each color. In addition, they showed that if m is even, then there exists a double wedge that contains exactly $m/2$ points of each color; a double wedge is the symmetric difference of two half-planes whose boundaries are not parallel.

Now, let P be a 3-colored point set of size n in general position in the plane, with n even. Assume the points in P are colored red, green, and blue such that P is color-balanced. Let R, G, and B denote the set of red, green, and blue points, respectively. Note that $|R|$, $|G|$, and $|B|$ are at most $\lfloor n/2 \rfloor$, but, they are not necessarily equal. Kano et al. [10] proved the existence of a feasible cut, when the points on the convex hull of P are monochromatic.