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Drawing the almost convex set in an integer grid of minimum size

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A R T I C L E I N F O

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In 2001, Károlyi, Pach and Tóth introduced a family of point sets to solve an Erdős–Szekeres type problem, which have been used to solve several other Erdős–Szekeres type problems. In this paper we refer to these sets as nested almost convex sets. A nested almost convex set $\mathcal{X}$ has the property that the interior of every triangle determined by three points in the same convex layer of $\mathcal{X}$, contains exactly one point of $\mathcal{X}$. In this paper, we introduce a characterization of nested almost convex sets. Our characterization implies that there exists at most one (up to equivalence of order type) nested almost convex set of n points, for any integer n. We use our characterization to obtain a linear time algorithm to construct nested almost convex sets of n points, with integer coordinates of absolute values at most $O(n^{\log_2 5})$. Finally, we use our characterization to obtain an $O(n \log n)$-time algorithm to determine whether a set of points is a nested almost convex set.

1. Introduction

We say that a set of points in the plane is in *general position* if no three of its points are collinear. Throughout this paper all points sets are in general position. In [9], Erdős asked for the minimum integer $E(s,l)$ that satisfies the following. Every set of at least $E(s,l)$ points, contains s points in convex position and at most l points in its interior. Let $\mathcal{X}$ be a set of points. A *k-hole* of $\mathcal{X}$ is a polygon with k vertices, all of which belong to $\mathcal{X}$ and without points of $\mathcal{X}$ in its interior; the polygon may be convex or non-convex. In 1983, Horton surprised the community with a simple proof that $E(s,l)$ does not exist for $l = 0$ and $s \geq 7$ [12]; Horton constructed arbitrarily large point sets with no convex 7-holes. Note that for $l = 0$, $E(s,l)$ is the minimum integer such that every set of at least $E(s,0)$ points contains at least one s-hole.

In 2001 [14] Károlyi, Pach and Tóth introduced a family of sets that have been used in various problems related to the original question of Erdős. They did not name this family; in this paper we refer to these sets as *nested almost convex sets*. They have been used in the following problems.

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A modular version of the Erdős problem. In 2001 [14] Károlyi, Pach and Tóth used the nested almost convex sets to prove that, for any $s \geq 5l/6 + O(1)$, there is an integer $B(s,l)$ with the following property. Every set of at least $B(s,l)$ points in general position contains s points in convex position such that the number of points in the interior of their convex hull is 0 modulo (l). This "modular" version of the Erdős problem was proved for $s \geq l+2$ by Bialostocki, Dierker, and Voxman [5]. The original upper bound on $B(s,l)$ was later improved by Caro in [7].

A version of the Erdős problem in almost convex sets. We say that $\mathcal{X}$ is an *almost convex* set if every triangle with vertices in $\mathcal{X}$ contains at most one point of $\mathcal{X}$ in its interior. Let $N(s)$ be the smallest integer such that every almost convex set of at least $N(s)$ points contains an s-hole. In 2007 [17] Valtr Lippner and Károlyi used the nested almost convex sets to prove that:

$$N(s) = \begin{cases} 2^{(s+1)/2} - 1 & \text{if } s \geq 3 \text{ is odd} \\ \frac{3}{2}2^{s/2} - 1 & \text{if } s \geq 4 \text{ is even.} \end{cases} \tag{1}$$

The authors used the nested almost convex sets to attain the equality in (1). The existence of $N(s)$ was first proved by Károlyi, Pach and Tóth in [14]. The upper bound for $N(s)$ was improved by Kun and Lippner in [15], and it was improved again by Valtr in [16].

Maximizing the number of non-convex 4-holes. In 2014 [1] Aichholzer, Fabila-Monroy, González-Aguilar, Hackl, Heredia, Huemer, Urrutia, and Vogtenhuber proved that the maximum number of non-convex 4-holes in a set of n points is at most $n^3/2 - \Theta(n^2)$. The authors used the nested almost convex sets to prove that some sets have $n^3/2 - \Theta(n^2 \log(n))$ non-convex 4-holes.

Blocking 5-holes. A set B blocks the convex k-holes in $\mathcal{X}$, if any k-hole of $\mathcal{X}$ contains at least one element of B in the interior of its convex hull. In 2015 [6] Cano, Garcia, Hurtado, Sakai, Tejel, and Urritia used the nested almost convex sets to prove that: $n/2 - 2$ points are always necessary and sometimes sufficient to block the 5-holes of a point set with n elements in convex position and $n = 4k$. The authors used the nested almost convex sets as an example of sets for which $n/2 - 2$ points are sufficient to block all 5-holes.

We now define formally the nested almost convex sets.

Definition 1.1. Let $\mathcal{X}$ be a point set; let k be the number of convex layers of $\mathcal{X}$; and for $1 \leq j \leq k$, let R_j be the set of points in the j-th convex layer of $\mathcal{X}$, with R_1 being the most internal one. We say that $\mathcal{X}$ is a nested almost convex set if:

1. $\mathcal{X}_j := R_1 \cup R_2 \cup \cdots \cup R_j$ is in general position,
2. the vertices in the boundary of the convex hull of $\mathcal{X}_j$ are the elements of R_j, and
3. any triangle determined by three points of R_j, with $j > 1$, contains precisely one point of $\mathcal{X}_{j-1}$ in its interior.

In this paper, we give a characterization of when a set of points is a nested almost convex set. This is done by first defining a family of trees. If there exists a map, satisfying certain properties, from the point set to the nodes of a tree in the family, then the point set is a nested almost convex set. This map encodes a lot of information about the point set. For example, it determines the location of any given point with respect to the convex hull; we use this information to obtain an $O(n \log n)$-time algorithm to decide whether a set of points is a nested almost convex set. This map also determines the orientation of any given triple of points. This implies that for every n there exists essentially at most one nested almost convex set. We further apply this information to obtain a linear-time algorithm that produces a representation of a nested almost convex set of n points (with $n = 2^{k-1} - 2$ or $n = 3 \cdot 2^{k-1} - 2$) on an small integer grid of size $O(n^{\log_2 5})$.

The *order type* of a point set $\mathcal{X} = \{x_1, x_2, \ldots x_n\}$ is a mapping that assigns to each ordered triple (x_i, x_j, x_k) an orientation. If x_k is to the left of the directed line from x_i to x_j, the orientation of (x_i, x_j, x_k) is counterclockwise. If x_k is to the right of the directed line from x_i to x_j, the orientation of (x_i, x_j, x_k) is clockwise. We say that two sets of points have the same order type, if there exists a bijection between these sets that preserves the orientation of all triples.

The order type was introduced by Goodman and Pollack in [10], and it has been widely used in Combinatorial Geometry to classify point sets; two sets of points are essentially the same if they have the same order type. As a consequence of the characterization of nested almost convex sets presented in Section 2, we have the following.

Theorem 1.2. *If $n = 2^{k-1} - 2$ or $n = 3 \cdot 2^{k-1} - 2$, there is exactly one order type that corresponds to a nested almost convex set with n points; for other values of n, nested almost convex sets with n points do not exist.*

In previous papers, two constructions of nested almost convex sets have been presented. The first construction was introduced by Károlyi, Pach and Tóth in [14]. The second construction was introduced by Valtr, Lippner and Károlyi in [17] six years later.

Construction 1: Let $\mathcal{X}_1$ be a set of two points. Assume that $j > 0$ and that $\mathcal{X}_j$ has been constructed. Let $z_1, \ldots, z_r$ denote the vertices of R_j in clockwise order. Let P_j be the polygon with vertices in R_j. Let $\varepsilon_j, \delta_j > 0$. For any $1 \leq i \leq r$,