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Flips in edge-labelled pseudo-triangulations ☆

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ABSTRACT

Given a set of n points in the plane, we show that $O(n^2)$ exchanging flips suffice to transform any edge-labelled pointed pseudo-triangulation into any other with the same set of labels. By using insertion, deletion and exchanging flips, we can transform any edge-labelled pseudo-triangulation into any other with $O(n \log c + h \log h)$ flips, where c is the number of convex layers and h is the number of points on the convex hull.

1. Introduction

A *pseudo-triangle* is a simple polygon with three convex interior angles, called *corners*, that are connected by reflex chains. Given a set P of n points in the plane, a *pseudo-triangulation* of P is a subdivision of its convex hull into pseudo-triangles, using all points of P as vertices (see Fig. 1a). A pseudo-triangulation is *pointed* if all vertices are incident to a reflex angle in some face (including the outer face; see Fig. 1b for an example). Pseudo-triangulations find applications in areas such as kinetic data structures [1] and rigidity theory [2]. More information on pseudo-triangulations can be found in a survey by Rote, Santos, and Streinu [3].

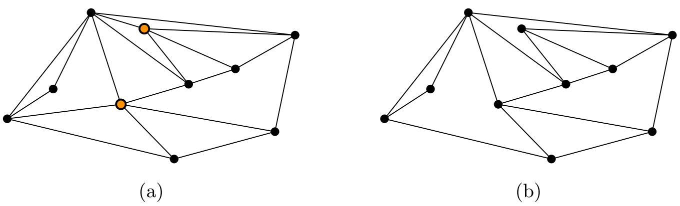

Fig. 1. (a) A pseudo-triangulation with two non-pointed vertices. (b) A pointed pseudo-triangulation.

Since a regular triangle is also a pseudo-triangle, pseudo-triangulations generalise triangulations (subdivisions of the convex hull into triangles). In a triangulation, a flip is a local transformation that removes one edge, leaving an empty

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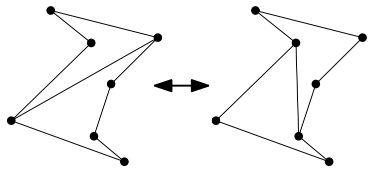

Fig. 2. A flip in a pseudo-quadrilateral.

quadrilateral, and inserts the other diagonal of that quadrilateral. Note that this is only allowed if the quadrilateral is convex, otherwise the resulting graph would be non-plane. Lawson [4] showed that any triangulation with n vertices can be transformed into any other with $O(n^2)$ flips, and Hurtado, Noy, and Urrutia [5] gave a matching $\Omega(n^2)$ lower bound.

Pointed pseudo-triangulations support a similar type of flip, but before we can introduce this, we need to generalise the concept of pseudo-triangles to *pseudo-k-gons*: weakly simple polygons[1] with k convex interior angles. We will typically refer to pseudo-4-gons and pseudo-5-gons as *pseudo-quadrilaterals* and *pseudo-pentagons*, respectively. A diagonal of a pseudo-k-gon is called a *bitangent* if the pseudo-k-gon remains pointed after insertion of the diagonal. In a pointed pseudo-triangulation, *flipping* an edge removes the edge, leaving a pseudo-quadrilateral, and inserts the unique other bitangent of the pseudo-quadrilateral (see Fig. 2). In contrast with triangulations, all internal edges of a pointed pseudo-triangulation are flippable. Bereg [6] showed that $O(n \log n)$ flips suffice to transform any pointed pseudo-triangulation into any other.

Aichholzer et al. [7] showed that the same result holds for all pseudo-triangulations (including triangulations) if we allow two more types of flips: *insertion* and *deletion* flips. As the name implies, these either insert or delete one edge, provided that the result is still a pseudo-triangulation. To disambiguate, they call the other flips *exchanging* flips. In a later paper, this bound was refined to $O(n \log c)$ [8], where c is the number of convex layers of the point set.

There is recent interest in *edge-labelled* triangulations: triangulations where each edge has a unique label, and flips reassign the label of the flipped edge to the new edge. Araujo-Pardo et al. [9] studied the flip graph of edge-labelled triangulations of a convex polygon, proving that it is still connected and in fact covers the regular flip graph. They also fully characterised its automorphism group. Independently, Bose et al. [10] showed that it has diameter $\Theta(n \log n)$.

Bose et al. also considered edge-labelled versions of other types of triangulations. In many cases, the flip graph is disconnected. For example, it is easy to create a triangulation on a set of points with an edge that can never be flipped. Two edge-labelled triangulations in which such an edge has different labels can therefore never be transformed into each other with flips. On the other hand, Bose et al. showed that the flip graph of edge-labelled combinatorial triangulations is connected and has diameter $\Theta(n \log n)$.

In this paper, we investigate flips in *edge-labelled pseudo-triangulations*: pseudo-triangulations where each internal edge has a unique label in $\{1, \ldots, 3n - 3 - 2h\}$, where h is the number of vertices on the convex hull ($3n - 3 - 2h$ is the number of internal edges in a triangulation). In the case of an exchanging flip, the new edge receives the label of the old edge. For a deletion flip, the edge and its label are removed, and for an insertion flip, the new edge receives an unused label from the set of all possible labels.

In contrast with the possibly disconnected flip graph of edge-labelled geometric triangulations, we show that $O(n^2)$ exchanging flips suffice to transform any edge-labelled pointed pseudo-triangulation into any other with the same set of labels. By using all three types of flips – insertion, deletion and exchanging – we can transform any edge-labelled pseudo-triangulation into any other with $O(n \log c + h \log h)$ flips, where c is the number of convex layers of the point set, and h is the number of vertices on the convex hull. In both settings, we have an $\Omega(n \log n)$ lower bound, making the second result tight in the worst case.

2. Pointed pseudo-triangulations

In this section, we show that every edge-labelled pointed pseudo-triangulation can be transformed into any other with the same set of labels by $O(n^2)$ exchanging flips. We do this by showing how to transform a given edge-labelled pointed pseudo-triangulation into a *canonical* one. The result then follows by the reversibility of flips.

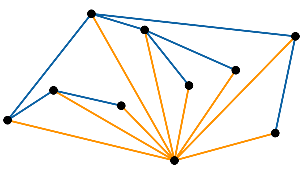

Fig. 3. A left-shelling pseudo-triangulation, with top and bottom edges highlighted.

[1] A weakly simple polygon is a plane graph with a bounded face that is incident to all edges.

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