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Two-level rectilinear Steiner trees

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ABSTRACT

Given a set P of terminals in the plane and a partition of P into k subsets $P_1, \dots, P_k$, a two-level rectilinear Steiner tree consists of a rectilinear Steiner tree T_i connecting the terminals in each set P_i $(i = 1, \dots, k)$ and a top-level tree T_{top} connecting the trees $T_1, \dots, T_k$. The goal is to minimize the total length of all trees. This problem arises naturally in the design of low-power physical implementations of parity functions on a computer chip.

For bounded k we present a polynomial time approximation scheme (PTAS) that is based on Arora's PTAS for rectilinear Steiner trees after lifting each partition into an extra dimension.

For the general case we propose an algorithm that predetermines a connection point for each T_i and T_{top} $(i = 1, \dots, k)$. Then, we apply any approximation algorithm for minimum rectilinear Steiner trees in the plane to compute each T_i and T_{top} independently.

This gives us a 2.37-factor approximation with a running time of $\mathcal{O}(|P| \log |P|)$ suitable for fast practical computations. The approximation factor reduces to 1.63 by applying Arora's approximation scheme in the plane.

1. Introduction

We consider the *two-level rectilinear Steiner tree problem* (R2STP) that arises from an application in VLSI design. Consider the computation of a parity function of k input bits using 2-input XOR-gates. Due to the symmetry, associativity, and commutativity of the XOR function, this can be realized by an arbitrary binary tree with k leaves, rooted at the output, by inserting an XOR-gate at every internal vertex (see Xiang et al. [16,17]). Throughout this paper we consider the parity function as a placeholder for any fan-in function of the type $x_1 \circ x_2 \circ \cdots \circ x_k$, where $\circ$ is a symmetric, associative, and commutative 2-input operator, i.e. $\circ \in \{\oplus, \vee, \wedge\}$.

On a chip such a tree has to be embedded into the plane and all connections must be realized by rectilinear segments. If each input and the output are single points on the chip, a realization of minimum length and, thus, power consumption is given by a minimum length rectilinear Steiner tree. This is a tree connecting the inputs and the output by horizontal and vertical line segments using additional so-called Steiner vertices to achieve a shorter length than a minimum spanning tree. At each Steiner vertex of degree three an XOR-gate is placed. Higher degree vertices can be dissolved into degree three vertices sharing their position. Fig. 1 shows an example of an embedded parity function on the left.

In practice, input signals may be needed for other computations on the chip and, thus, delivered to other side outputs. Similarly, the result may have to be delivered to multiple output terminals. Thus, each input and its successors and the

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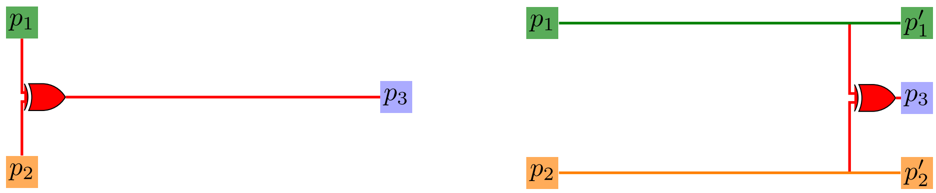

Fig. 1. On the left, we have two inputs p_1 and p_2 and a single output p_3. The XOR-gate should be placed at the median of the three terminals. If the inputs have the side outputs p_1' and p_2', the XOR-gate should be placed at p_3, saving the horizontal length.

output terminals must be connected by separate Steiner trees as well. These trees are then connected by a top-level Steiner tree into which the XOR-gates will be inserted. Considering the additional terminals allows to construct a potentially shorter top-level and two-level Steiner tree as shown in Fig. 1 on the right. Algorithms ignoring the side outputs cannot guarantee an approximation factor better than two, as we will see in Section 2.

This motivates the definition of the *minimum two-level rectilinear Steiner tree problem*, where we are given a set $P \subset \mathbb{R}^2$ of n terminals and a partition of P into k subsets $P_1, \dots, P_k$.

A *two-level rectilinear Steiner tree* $T = (T_{top}, T_1, \dots, T_k)$ consists of a Steiner tree T_i for each $i \in \{1, \dots, k\}$ connecting the terminals in P_i and a (group) Steiner tree T_{top} connecting the embedded trees $\{T_1, \dots, T_k\}$. We call T_{top} the *top-level tree*. Note that all trees are allowed to cross. The objective is to minimize the total length of all trees

$$l(T) := \sum_{i=1}^{k} l(T_i) + l(T_{top}),$$

where $l(T') := \sum_{\{x,y\} \in E(T')} \|x - y\|_1$ is the ℓ_1-length of a Steiner tree T'.

For each $i \in \{1, \dots, k\}$ the top-level tree and T_i intersect in at least one point. We can select one such point $q_i \in T_{top} \cap T_i$ and call it *connection point* for T_i and T_{top}. Then T_{top} is a Steiner tree for the terminals $\{q_1, \dots, q_k\}$ and each T_i is a Steiner tree for $P_i \cup \{q_i\}$.

For a compact set $Q \subset \mathbb{R}^d$, we denote by $B(Q)$ the (minimum axis-parallel) *bounding box* containing Q and by $l(B(Q))$ the *bounding box length of* Q, i.e. the sum of widths of $B(Q)$ over all d dimensions. Using the following simple observation, we can always assume that connection points are located in the bounding box of their partition: $q_i \in B(P_i)$ $(1 \le i \le k)$.

Observation 1.1. *Given a two-level Steiner tree $T = (T_{top}, T_1, \dots, T_k)$ for $P_1, \dots, P_k$, we can always find a two-level Steiner tree $T' = (T'_{top}, T'_1, \dots, T'_k)$ for $P_1, \dots, P_k$ and connection points $q'_i \in B(P_i)$ so that $l(T') \le l(T)$ and $l(T'_i) \le l(T_i)$ for all $i = 1, \dots, k$.*

Proof. Consider a tree T_i, $i \in \{1, \dots, k\}$, for which $q_i \notin B(P_i)$. We insert a new vertex q'_i into T_i at the same position as q_i, reconnect vertices in $V(T_i)$ that are adjacent to q_i to q'_i, and add a new edge $\{q_i, q'_i\}$. Furthermore, we subdivide any edge $\{v, w\} \in E(T_i)$ crossing the boundary $\partial B(P_i)$, by a new vertex placed in the intersection $B(\{v, w\}) \cap \partial B(P_i)$ (if an edge is crossing two edges of $\partial B(\{v, w\})$, it will be subdivided twice). This does not alter any tree length. Now, we project all vertices of $T_i - q_i$ which are not inside the bounding box $B(P_i)$ to the closest point on its boundary. Thereby, q'_i is relocated to the boundary of $B(P_i)$ with minimum distance to q_i. The length of T_i does not increase, because the total length of the edges incident to projected vertices must have been attained at least by the edges outside $B(P_i)$ before the projection.

Finally, let q'_i be the new connection point, $T'_{top} = T_{top} + \{q_i, q'_i\}$, and $T'_i = T_i - \{q_i, q'_i\}$. This results in a solution T' with $l(T') \le l(T)$ and $l(T'_i) \le l(T_i)$. $\square$

Obviously, the rectilinear two-level Steiner tree problem is *NP*-hard as it contains the minimum rectilinear Steiner tree problem in two ways: if $k = 1$ or if $|P_i| = 1$ for $i \in \{1, \dots, k\}$. The problem is *NP*-complete, because there is always an optimum solution in the Hanan grid [9] of P. This simple fact will arise later as a side-result in Corollaries 3.1 and 3.2.

Designing the top-level tree as a stand-alone problem is hard. If all subtrees T_i $(i \in \{1, \dots, k\})$ are fixed, T_{top} cannot be approximated to arbitrary quality, as the group Steiner tree problem for connected groups in the Euclidean plane cannot be approximated within a factor of $(2 - \epsilon)$ due to Safra and Schwartz [12]. However, we are in a more lucky situation as we can tradeoff the lengths of bottom-level and top-level trees.

To the best of our knowledge the two-level rectilinear Steiner tree problem has not been considered before despite its practical importance [16,17]. It is, however, a specialization of the clustered Steiner tree problem introduced by Wu and Lin [15]. Here, a solution consists of a Steiner tree T for P that contains a subtree T_i connecting P_i for each $i = 1, \dots, k$, where all k subtrees are pairwise disjoint. Wu and Lin present an $(\alpha + 2)$-factor approximation algorithm for the clustered Steiner tree problem in metric spaces, given an α-factor approximation for the Steiner tree problem. A two-level Steiner tree is a clustered Steiner tree but not vice versa, as a connected top-level tree need not exist in clustered Steiner trees. Thus, approximation results cannot be transferred easily. The two-level Steiner tree problem is also loosely related to the hierarchical network design problems studied by Alvarez-Miranda et al. [1], Balakrishnan, Magnanti, and Mirchandani [4], and Current, ReVelle, and Cohon [7]. There is also some similarity to multi-level facility location problems studied by