Discrete Applied Mathematics 233 (2017) 29–32

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Upper bounds on adjacent vertex distinguishing total chromatic number of graphs

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ARTICLE INFO

Article history:
Received 29 March 2016
Received in revised form 22 July 2017
Accepted 14 August 2017
Available online 21 September 2017

Keywords:
Adjacent vertex distinguishing total coloring
Chromatic number
Edge chromatic number
Maximum degree

ABSTRACT

An adjacent vertex distinguishing total coloring of a graph G is a proper total coloring of G such that for any pair of adjacent vertices, the set of colors appearing on the vertex and incident edges are different. The minimum number of colors required for an adjacent vertex distinguishing total coloring of G is denoted by $\chi''_a(G)$. Let G be a graph, and $\chi(G)$ and $\chi'(G)$ be the chromatic number and edge chromatic number of G, respectively. In this paper we show that $\chi''_a(G) \leq \chi(G) + \chi'(G) - 1$ for any graph G with $\chi(G) \geq 6$, and $\chi''_a(G) \leq \chi(G) + \Delta(G)$ for any graph G. Our results improve the only known upper bound 2Δ obtained by Huang et al. (2012). As a direct consequence, we have $\chi''_a(G) \leq \Delta(G) + 3$ if $\chi(G) = 3$ and thus it implies the known results on graphs with maximum degree 3, K_4-minor-free graphs, outerplanar graphs, graphs with maximum average degree less than 3, planar graphs with girth at least 4 and 2-degenerate graphs.

1. Introduction

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. We use $N_G(v)$ to denote the neighbors of a vertex v in G and $d_G(v) = |N_G(v)|$. Let $\Delta(G)$ denote the *maximum degree* of G. If there is no confusion from the context we use simply Δ.

A *proper k-coloring* is a mapping $\phi : V(G) \longrightarrow \{1, 2, \ldots, k\}$ such that any two adjacent vertices in $V(G)$ receive different colors. The *chromatic number* $\chi(G)$ of G is the smallest integer k such that G has a proper k-coloring. A *proper k-edge coloring* is a mapping $\phi : E(G) \longrightarrow \{1, 2, \ldots, k\}$ such that any two adjacent edges in $E(G)$ receive different colors. The *edge chromatic number* $\chi'(G)$ of G is the smallest integer k such that G has a proper k-edge coloring. A *proper total k-coloring* is a mapping $\phi : V(G) \cup E(G) \longrightarrow \{1, 2, \ldots, k\}$ such that any two adjacent or incident elements in $V(G) \cup E(G)$ receive different colors. The *total chromatic number* $\chi''(G)$ of G is the smallest integer k such that G has a proper total k-coloring. Let ϕ be a proper total coloring of G. Denote $C_\phi(v) = \{\phi(uv) \mid uv \in E(G)\} \cup \{\phi(v)\}$. A proper total k-coloring ϕ of G is *adjacent vertex distinguishing*, or a *total-k-avd-coloring*, if $C_\phi(u) \neq C_\phi(v)$ whenever $uv \in E(G)$. The *adjacent vertex distinguishing total chromatic number* $\chi''_a(G)$ is the smallest integer k such that G has a total-k-avd-coloring. It is obvious that $\Delta + 1 \leq \chi''(G) \leq \chi''_a(G)$.

In [13], Zhang et al. introduced the notation of adjacent vertex distinguishing total colorings of graphs and made the following conjecture in terms of the maximum degree Δ.

Conjecture 1.1. *For any graph G, $\chi''_a(G) \leq \Delta + 3$.*

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http://dx.doi.org/10.1016/j.dam.2017.08.016
0166-218X/

Chen [2] and Wang [8], independently, confirmed Conjecture 1.1 for graphs G with $\Delta \leq 3$. Later, Hulgan [5] presented a more concise proof on this result. Wang et al. considered the adjacent vertex distinguishing total chromatic number of K_4-minor-free graphs [11] and outerplanar graphs [12]. In [10], Wang considered the adjacent vertex distinguishing total chromatic number of sparse graphs. Wang and Huang [9] considered the adjacent vertex distinguishing total colorings of planar graphs with $\Delta \geq 11$. In [6], Miao et al. considered the adjacent vertex distinguishing total chromatic number of 2-degenerate graphs.

For general graphs, Huang et al. [4] proved the following upper bound.

Theorem 1.2. *Let G be a graph with $\Delta \geq 3$, then $\chi''_a(G) \leq 2\Delta$.*

By definitions of coloring, edge coloring and adjacent vertex distinguishing total coloring, we have the following proposition which is also observed in [4]:

Proposition 1.3. *For any graph G, $\chi''_a(G) \leq \chi(G) + \chi'(G)$.*

In this paper, by applying recoloring techniques, we prove a new upper bound for $\chi''_a(G)$ for general graphs.

Theorem 1.4. *Let G be a graph with $\chi(G) \geq 6$. Then $\chi''_a(G) \leq \chi(G) + \chi'(G) - 1$.*

In this paper we first present a new upper bound on $\chi''_a(G)$.

Theorem 1.5. *For each graph G, $\chi''_a(G) \leq \chi(G) + \Delta$.*

Theorem 1.5 implies the upper bound 2Δ as stated in Theorem 1.2 as well as the case when $\Delta = 3$. By Four Color Theorem [1], it implies that for any planar graph G, $\chi''_a(G) \leq \Delta + 4$ while the conjecture is $\chi''_a(G) \leq \Delta + 3$. It also implies that Conjecture 1.1 is true for all 3-colorable graphs.

Corollary 1.6. *If G is 3-colorable, then $\chi''_a(G) \leq \Delta + 3$.*

Note that the family of 3-colorable graphs includes many interesting subfamilies of graphs such as graphs with maximum degree 3 (except K_4), K_4-minor-free graphs, outerplanar graphs, graphs with maximum average degree less than 3, and 2-degenerate graphs. Therefore Corollary 1.6 implies the known results for those families of graphs. Also by the well-known Grötzsch's 3-color theorem [3], we have the following result which is a special case of Corollary 1.6.

Corollary 1.7. *Let G be a triangle-free planar graph. Then $\chi''_a(G) \leq \Delta + 3$ and thus Conjecture 1.1 is true for all triangle-free planar graphs.*

2. Proof of Theorem 1.5

Proof. Denote $k = \chi(G)$. Let $\phi : V(G) \to \{\Delta+1, \Delta+2, \dots, \Delta+k\}$ be a proper vertex coloring of G. By Vizing's Theorem [7], every graph G is $(\Delta + 1)$-edge coloring. Let $\psi : E(G) \to \{1, 2, \dots, \Delta + 1\}$ be a proper edge coloring of G. Then ϕ together with ψ is a total coloring which may not be proper since some edges colored with $\Delta + 1$ may be incident with a vertex colored with $\Delta + 1$ too.

Let A be the set of all edges which are colored with $\Delta + 1$ and are incident with a vertex colored with $\Delta + 1$. That is

$$A = \{uv \in E(G) \mid \psi(uv) = \Delta + 1 \in \{\phi(u), \phi(v)\}\}.$$

Then A is a matching since ψ is a proper edge coloring.

Denote $V_1 = \{u \mid \phi(u) = \Delta + 1$, and u is incident with an edge in $A\}$. The V_1 is an independent set.

Let $u \in V_1$ with $uv \in A$. Since $\psi(uv) = \Delta + 1$ and $d(u) \leq \Delta$, there exists a color $\alpha \in \{1, 2, \dots, \Delta\} \setminus \{\psi(ux) \mid x \in N(u)\}$. Recolor u with α. Let φ be the new total coloring after applying the recoloring on all the vertices in V_1. Noting that the colors of the edges are unchanged and in $\{1, 2, \dots, \Delta + 1\}$, and the colors of the vertices not in V_1 are unchanged and in $\{\Delta + 1, \Delta + 2, \dots, \Delta + k\}$, we have φ is a $(\Delta + k)$-proper total coloring. In the following, we show that φ is an adjacent vertex distinguishing $(\Delta + k)$-total coloring.

Let xy be an edge in G. Since V_1 is an independent set, assume without loss of generality that $y \notin V_1$ and $\varphi(y) = \Delta + j$ for some $j \geq 1$.

If $x \in V_1$, then $\phi(x) = \Delta + 1$ and thus $\phi(y) = \Delta + j \geq \Delta + 2$. Now in the coloring φ, $\varphi(x) < \Delta + 1$. And thus $C_\varphi(x) \subseteq \{1, 2, \dots, \Delta+1\}$ since the colors of all edges are unchanged and are in $\{1, 2, \dots, \Delta+1\}$. Since $\Delta+j = \varphi(y) \in C_\varphi(y)$, then $C_\varphi(x) \neq C_\varphi(y)$.

If $x \notin V_1$, then $\varphi(u) = \phi(u) = \Delta + i$ for some $i \geq 1$ and $i \neq j$. We may assume without loss of generality that $j \geq 2$. In this case $C_\varphi(x) = C_\phi(x)$ and $C_\varphi(y) = C_\phi(y)$. Since all the edges incident with x or y are colored with colors in $\{1, 2, \dots, \Delta + 1\}$, then $\Delta + j \notin C_\varphi(x)$, and thus $C_\varphi(x) \neq C_\varphi(y)$.

Therefore, we show that $C_\varphi(x) \neq C_\varphi(y)$ for any $xy \in E(G)$, i.e., φ is a total-$(k + \Delta)$-avd-coloring and therefore completes the proof of Theorem 1.5. ■