ARTICLE IN PRESS

Discrete Applied Mathematics ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

# Minimum sum coloring problem: Upper bounds for the chromatic strength

Clément Lecat *, Corinne Lucet, Chu-Min Li

*Laboratoire de Modélisation, Information et Système EA 4290, Université de Picardie Jules Verne, Amiens, France*

ARTICLE INFO

*Article history:*
Received 14 February 2017
Received in revised form 4 July 2017
Accepted 20 July 2017
Available online xxxx

*Keywords:*
Graph theory
Sum coloring problem
Chromatic strength

ABSTRACT

The minimum sum coloring problem (*MSCP*) is a vertex coloring problem in which a weight is associated with each color. Its aim is to find a coloring of the vertices of a graph $G$ with the minimum sum of the weights of the used colors. The *MSCP* has important applications in the fields such as scheduling and VLSI design. The minimum number of colors among all optimal solutions of the *MSCP* for $G$ is called the *chromatic strength* of $G$ and is denoted by $s(G)$. A tight upper bound of $s(G)$ allows to significantly reduce the search space when solving the *MSCP*. In this paper, we propose and empirically evaluate two new upper bounds of $s(G)$ for general graphs, $UB_A$ and $UB_S$, based on an abstraction of all possible colorings of $G$ formulated as an ordered set of decreasing positive integer sequences. The experimental results on the standard benchmarks DIMACS and COLOR show that $UB_A$ is competitive, and that $UB_S$ is significantly tighter than those previously proposed in the literature for 70 graphs out of 74 and, in particular, reaches optimality for 8 graphs. Moreover, both $UB_A$ and $UB_S$ can be applied to the more general optimum cost chromatic partition (*OCCP*) problem.



## 1. Introduction

The optimum cost chromatic partition (*OCCP*) problem [27] consists in finding a valid vertex coloring of a graph $G = (V, E)$ with some colors $c_1, c_2, \dots, c_k$, taking into account their associated weights $w_1, w_2, \dots, w_k$, such that the sum of the color weights is minimum. This problem has many applications in VLSI when restricted to circle and permutation graphs [31], in scheduling problems for interval graphs [19] and in resource allocation [1,25].

When $(w_1, w_2, \dots, w_k) = (1, 2, \dots, k)$, the *OCCP* problem becomes the minimum sum coloring problem (*MSCP*). The sum of the color weights in an optimal solution of the *MSCP* is denoted by $\Sigma(G)$. The *MSCP* was introduced in 1989 and shown to be NP-hard by Kubicka and Schwenk [20]. Several optimal solutions may exist for a graph $G$. These optimal solutions give the same $\Sigma(G)$ but do not necessarily use the same number of colors. The minimum number of colors among all optimal solutions of the *MSCP* for a graph $G$ is called the *chromatic strength* of $G$ and is denoted by $s(G)$. Note that $s(G)$ can be bigger than the minimum number $\chi(G)$ of colors required to color $G$, which is called the *chromatic number* of $G$.

We find in the literature many theoretical results proving theoretical bounds of $\Sigma(G)$ or $s(G)$ [2,13,25,33] and structural properties relative to the graph families for which polynomial algorithms exist [13,19], and a large number of approximation algorithms to solve the *MSCP* for specific graph families (see, e.g., [9,10]). We also find in the literature several heuristic methods, such as the greedy algorithms *MDSAT* and *MRLF* [24], and meta-heuristic algorithms [3,17,28,32] based on evolutionary and tabu search to solve the *MSCP* for general graphs. All of these methods were tested on DIMACS and COLOR

* Corresponding author.
*E-mail addresses:* clement.lecat@u-picardie.fr (C. Lecat), corinne.lucet@u-picardie.fr (C. Lucet), chu-min.li@u-picardie.fr (C. Li).

http://dx.doi.org/10.1016/j.dam.2017.07.025

benchmarks [7,15]. Some exact methods were also proposed in [21] based on the classical branch-and-bound scheme or on the minimum/maximum propositional satisfiability (MinSAT/MaxSAT).

When solving the *MSCP* for a general graph $G$, the algorithms have to explore the search space, which grows exponentially with the number of colors to be considered. In practice, it is substantially more difficult to reduce this number when solving the *MSCP* than when solving the classical graph coloring problem (*GCP*). In fact, when a valid coloring solution with $k$ colors is found, the sub-space with $k$ or more colors can be pruned for the *GCP* because an optimal solution for the *GCP* cannot contain more than $k$ colors in this case. However, an optimal solution of the *MSCP* can involve more than $k$ colors. Therefore, the sub-space with $k$ or more colors cannot be pruned because of the valid coloring solution with $k$ colors. Consequently, establishing a tight upper bound of $s(G)$ is essential for solving the *MSCP*.

Different tight bounds of $s(G)$ have been established in the literature for special graph families, such as trees, interval graphs or partial $k$-trees [13,20,27,29]. An upper bound of $s(G)$ for planar graphs can be derived from a result in [9,10] for the sum multi-coloring problem on planar graphs. Unfortunately, to the best of our knowledge, only two upper bounds of $s(G)$ have been proposed for a general graph $G$ in [27] and in [8].

In this paper, we propose two new upper bounds of $s(G)$ for a general graph $G$, called $UB_A$ and $UB_S$. We derive these upper bounds by exploring an abstraction of the set of coloring solutions of $G$ called *motifs*. Roughly speaking, a motif is a decreasing sequence of positive integers in which each integer represents the number of vertices in $G$ assigned a color $c$. The notion of motifs was already used in [4,5] to solve the *MSCP* for $P_4$-sparse graphs and in [22] to compute a lower bound of $\Sigma(G)$ in the general case. However, to the best of our knowledge, it is the first time that motifs are used for upper bounding $s(G)$. By skilfully identifying and excluding those motifs that cannot correspond to an optimal solution of the *MSCP*, we derive the two new upper bounds of $s(G)$ from the remaining motifs. The experimental results on the standard benchmarks DIMACS and COLOR [6,12] for coloring problems show that $UB_A$ is competitive and $UB_S$ is substantially better, in general, than the upper bounds proposed in [8,27]. In particular, $UB_S$ gives and proves the optimal value of the chromatic strength for eight graphs.

This paper is organized as follows. Section 2 presents the definitions and some well-known graph problems. It also introduces the notions of major colorings and motifs. Section 3 presents the previous upper bounds of $s(G)$ proposed in the literature and a number of properties of the motifs that allow the new upper bounds $UB_A$ and $UB_S$ of $s(G)$ to be established. Section 4 empirically evaluates $UB_A$ and $UB_S$ by comparing them with the previous upper bounds of $s(G)$ on the DIMACS and COLOR graphs. Section 5 concludes this paper.

## 2. Definitions and formulations

### 2.1. Chromatic number, minimum sum coloring, maximum clique and maximum stable set

We consider an undirected graph $G = (V, E)$, where $V$ is a set of vertices ($|V| = n$) and $E \subseteq V^2$ is a set of edges. The set of adjacent (or neighbor) vertices of $v \in V$, denoted by $\mathcal{N}$, is defined as $\mathcal{N}(v) = \{u \mid (u, v) \in E\}$. The degree $d(v)$ of a vertex $v$ is the number of its adjacent vertices, i.e., $d(v) = |\mathcal{N}(v)|$. The degree of a graph, denoted by $\Delta(G)$, is $max\{d(v) \mid v \in V\}$.

A clique $C$ is a subset of $V$ such that $\forall u, v \in C, (u, v) \in E$. The maximum clique (MaxClique) problem consists in finding a clique with the maximum cardinality in $G$.

A stable set $S$ is a subset of $V$ such that $\forall u, v \in S, (u, v) \notin E$. The maximum stable set (MaxStable) problem consists in finding a stable set with the maximum cardinality in $G$.

The complement graph of $G$ is defined as $\overline{G} = (V, \overline{E})$, where $\overline{E} = V^2 \setminus E$. Note that a clique in $G$ is a stable set in $\overline{G}$ and vice versa.

A coloring of a graph $G$ with $k$ colors is a function $c : V \mapsto \{1, 2, \ldots, k\}$ that assigns to each vertex $v \in V$ a color $c(v)$. A coloring is valid if $\forall(u, v) \in E, c(u) \neq c(v)$. We denote a valid coloring of $G$ with $k$ colors by $X = \{X_1, X_2, \ldots, X_k\}$, where $X_i = \{v \in V \mid c(v) = i\}$ is called a color class, that is, a stable set of $G$. The graph coloring problem (*GCP*) consists in finding a valid coloring $X$ of $G$ with minimum $k$. This $k$ is called the *chromatic number* of $G$, and is denoted by $\chi(G)$. The *GCP* is NP-hard [18].

The *MSCP* associates a weight $w_i = i$ with each color $i$. We denote $\Sigma(X)$ as the sum of color weights of a coloring $X$:

$$\Sigma(X) = 1 \times |X_1| + 2 \times |X_2| + \cdots + k \times |X_k|.$$

**Example 1.** Referring to the graph in Fig. 1, $X = \{\{a, e\}_1, \{b\}_2, \{c, d, f\}_3\}$ is a valid coloring. The vertices $a$ and $e$ are colored with color 1, the vertex $b$ with color 2, and the vertices $c$, $d$ and $f$ with color 3. The sum coloring is $\Sigma(X) = 1\times2+2\times1+3\times3 = 13$.

Given a graph $G$, the *MSCP* consists in finding a valid coloring $X$ of $G$ with the minimum sum of color weights. This minimum sum is called the chromatic sum of $G$ and is denoted by $\Sigma(G)$:

$$\Sigma(G) = \min\{\Sigma(X) \mid X \textit{ is a valid coloring of } G\}$$

Kubicka and Schwenk proved that the *MSCP* is NP-hard [20]. An optimal solution of the *GCP* does not necessarily correspond to an optimal solution of the *MSCP*. For example, an optimal solution of the *GCP* for the graph in Fig. 2 uses 2 colors and $\Sigma(X)$ =12, while an optimal solution of the *MSCP* for this graph uses 3 colors. The chromatic sum of the graph is