Discrete Applied Mathematics 233 (2017) 143–158

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

The number of completely different optimal identifying codes in the infinite square grid

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ARTICLE INFO

Article history:
Received 24 February 2016
Received in revised form 26 June 2017
Accepted 4 July 2017
Available online 4 August 2017

Keywords:
Identifying code
Difference
Lattice
Density
Discrete geometry

ABSTRACT

Let G be a graph with vertex set V and edge set E. We call any subset $C \subseteq V$ an identifying code if the sets

$$I(v) = \{c \in C \mid \{c, v\} \in E \text{ or } c = v\}$$

are distinct and non-empty for all vertices $v \in V$. We study identifying codes in the infinite square grid. The vertex set of this graph is $\mathbb{Z}^2$ and two vertices are connected by an edge if the Euclidean distance between these vertices is one. Ben-Haim & Litsyn have proved that the minimum density of identifying code in the infinite square grid is $\frac{7}{20}$. Such codes are called optimal. We study the number of completely different optimal identifying codes in the infinite square grid. Two codes are called completely different if there exists an integer n such that no $n \times n$-square of one code is equivalent with any $n \times n$-square of the other code. In particular, we show that there are exactly two completely different optimal periodic codes and no optimal identifying code is completely different with both of these two periodic codes.

1. Introduction

This paper is concerned with the number of essentially different identifying codes in the infinite square grid. It is shown in [1] and [2] that there are (at least) two different optimal periodic identifying codes in the infinite square grid. See Fig. 1. In this paper, we shall show that every optimal identifying code in the infinite square grid contains an arbitrary large area which is equivalent with one of these two identifying codes. Moreover, we show that the only periodic optimal identifying codes are given in Fig. 1.

First of all, we define basic notations.

Identifying codes are defined in a given graph $G = (V, E)$. For all vertices $v \in V$, the set $B(v) = \{u \in V \mid (u, v) \in E \text{ or } u = v\}$ is called the *neighbourhood* of v.

Any subset C of vertices V is called a *code* and a vertex in the code is called a *codeword*. A code C is called an *identifying code* if the *identifying sets*

$$I(v) = C \cap B(v)$$

are non-empty and distinct for all vertices $v \in V$. An identifying code with the minimum number of vertices is called *optimal*.

In this paper, we nevertheless study identifying codes in infinite graphs — exactly in the infinite square grid. Therefore, we need to define the number of codewords with the proportion of codewords to all vertices, that is with density. We first

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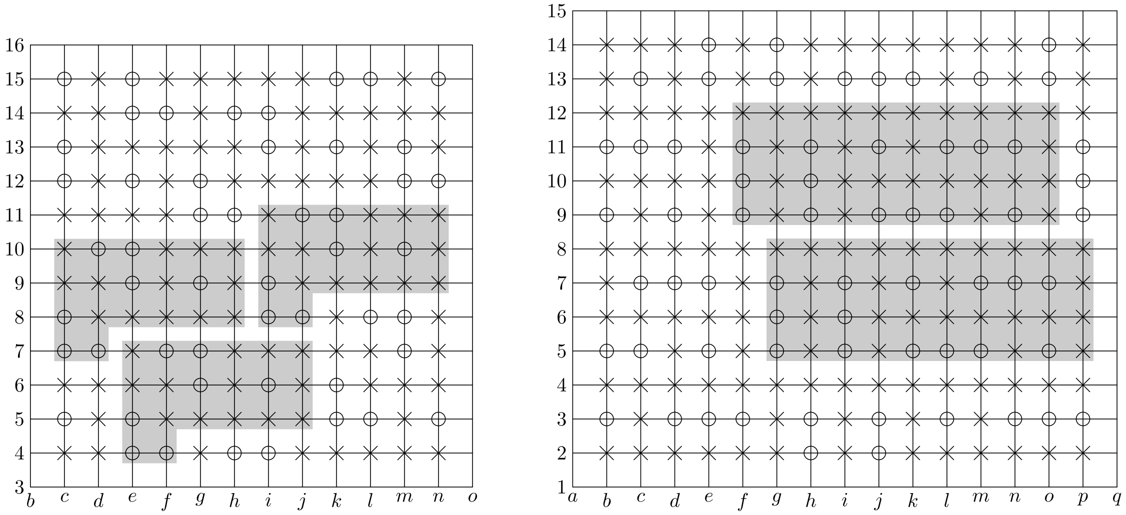
http://dx.doi.org/10.1016/j.dam.2017.07.012



(a) C_1. (b) C_2.

Fig. 1. A part of the only optimal periodic identifying codes. Circles are codewords and crosses are non-codewords. The periodic codes consist of similar tiles which are illustrated by grey backgrounds.

define the *infinite square grid*. the vertex set of this graph is $\mathbb{Z}^2 = \{(x, y) | x \in \mathbb{Z}, y \in \mathbb{Z}\}$ and two vertices are connected by an edge if the Euclidean distance between the vertices is one. The *density* of code C is

$$D(C) = \limsup_{n\to\infty} \frac{|C \cap Q_n|}{|Q_n|},$$

where $Q_n = \{(x, y) : -n \le x \le n, -n \le y \le n\}$. Now, a code is called *optimal* if the density is minimum.

Identifying codes were introduced in [3] in the 1990s. A motivation for such codes is a safeguard analysis of a facility using sensor networks [5]. Assume that we want to detect motion in a facility and we can put some detectors against thieves, fire, etc. to the rooms of the facility. Assume also that a detector gives an alarm if it detects motion in the room where the detector itself is or in the nearby rooms. Now, we want to place the detectors in such a way that some detector always alarms if there is motion in the facility. Moreover, we also want to know in which room the motion is, based only on the knowledge which detectors observed the motion. Now, if we place the detectors in such a way that they form an identifying code, then this is possible.

2. A definition for completely different codes

Next we study how we count the number of essentially different optimal identifying codes. The same definition has also been used in [4].

The problem in relation to the number of essentially different codes is that two codes are different if they differ in only finite many vertices. However, the finite many codewords do not influence the density of code. For example, if C is an optimal code, then $C' = C \cup A$, where A is any finite set of vertices, is also an optimal code.

Therefore, we need a stronger definition when two codes are essentially different. We define it by completely different codes. Before the definition, we define the isometry in $\mathbb{Z}^2$.

Definition 1. A mapping $\alpha : \mathbb{Z}^2 \to \mathbb{Z}^2$ is called an *isometry* in $\mathbb{Z}^2$ if it maps vertices to vertices bijectively and preserves the edges and the Euclidean distances between all vertices. The isometries consist of translations, reflections, rotations and glide reflections.

Definition 2. Two codes C_a and C_b are called *completely different* if there exists a positive integer n such that

$$\alpha(C_a) \cap Q_n \neq \beta(C_b) \cap Q_n$$

for all isometries α and β.

In other words, two codes C_a and C_b are not completely different if there are arbitrary large squares in C_a and C_b, which are similar.

This definition gives an interesting way to count the number of optimal codes. Indeed, for any optimal code in the given graph, we can often find a property with the help of which we can define how many codes exist in the given graph according