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Distant irregularity strength of graphs with bounded minimum degree

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ABSTRACT

Consider a graph $G=(V,E)$ without isolated edges and with maximum degree Δ. Given a colouring $c: E\to\{1,2,\ldots,k\}$, the weighted degree of a vertex $v\in V$ is the sum of its incident colours, i.e., $\sum_{e\ni v}c(e)$. For any integer $r\geq 2$, the least k admitting the existence of such c attributing distinct weighted degrees to any two different vertices at distance at most r in G is called the r-distant irregularity strength of G and denoted by $s_r(G)$. This graph invariant provides a natural link between the well known 1–2–3 Conjecture and irregularity strength of graphs. In this paper we apply the probabilistic method in order to prove an upper bound $s_r(G)\leq(4+o(1))\Delta^{r-1}$ for graphs with minimum degree $\delta\geq\ln^8\Delta$, improving thus far best upper bound $s_r(G)\leq 6\Delta^{r-1}$.

1. Introduction

Let us consider a graph $G=(V,E)$ and its not necessarily proper edge colouring $c: E\to\{1,2,\ldots,k\}$ with k least positive integers. We say that such colouring c is *irregular* if it associates with every vertex $v\in V$ a different sum of its incident colours:

$$w_c(v) := \sum_{u\in N(v)} c(uv), \tag{1}$$

so called *weighted degree* of v. We shall also denote $w_c(v)$ by $w(v)$ in cases when the colouring c is unambiguous from context. The least k admitting such irregular colouring c is called the *irregularity strength* of G and denoted by $s(G)$, see [7]. Note that this parameter is well defined for graphs without isolated edges and with at most one isolated vertex; for the remaining ones we might e.g. set $s(G)=\infty$. Alternatively, $s(G)$ might be regarded as the least k so that we may construct an irregular multigraph, i.e. a multigraph with pairwise distinct degrees of all vertices, of G by multiplying its edges, each at most k times. This study thus originate from the basic fact that no graph G with more than one vertex is irregular itself, hence $s(G)\geq 2$, and related research on possible alternative definitions of irregularity in graph environment, see e.g. [6]. It is known that $s(G)\leq n-1$, where $n=|V|$, for all graphs containing no isolated edges and at most one isolated vertex, except for the graph K_3, see [3,20]. This is a tight upper bound, as exemplified e.g. by the family of stars. A better upper bound is known for graphs with minimum degree $\delta>6$, i.e., $s(G)\leq 6\lceil\frac{n}{\delta}\rceil$, see [15], and $s(G)\leq(4+o(1))\frac{n}{\delta}+4$ for graphs with $\delta\geq n^{0.5}\ln n$, see [18]. It is however believed that these upper bounds can be improved to (in such a case optimal) $s(G)\leq\frac{n}{\delta}+C$ for some absolute constant C, see e.g. [11,15,18,23]. This has been explicitly conjectured in the case of d-regular graphs, see [11], for which one can observe that on the other hand $s(G)\geq\frac{n}{d}+\frac{d-1}{d}$ via straightforward counting argument, see e.g. [7]. Other results concerning the concept of irregularity strength and in particular its value for specific graph classes can also be found e.g. in [5,8–10,12,17,22], and many others.

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The problem described above gave rise to a variety of associated questions and concepts, nowadays making up an intensively studied field of the graph theory. One of its most intriguing descendants is its local version, where one investigates the least k for which there is a colouring $c : E \to \{1, 2, \dots, k\}$ of the edges of a graph G such that $w_c(u) \neq w_c(v)$ for every pair of adjacent vertices u, v, so called *neighbours* in G. Denote this value by $s_1(G)$ (we shall comment on this notion below), and note that such graph invariant is well defined for all graphs without isolated edges. Though initially no finite upper bound was known for this parameter, Karoński, Łuczak and Thomason [16] posed a fascinating conjecture that $s_1(G) \leq 3$ for all graphs without isolated edges. This is nowadays commonly referred to as *1–2–3 Conjecture* in the literature, see e.g. [14]. The conjecture is still open, while thus far the following general upper bounds were subsequently proved: $s_1(G) \leq 30$ in [1], $s_1(G) \leq 16$ in [2], $s_1(G) \leq 13$ in [25], and finally $s_1(G) \leq 5$ from [14].

In this paper we study a problem linking the two concepts above. Given any graph $G = (V, E)$ and an integer $r \geq 1$, two distinct vertices at distance at most r in G, i.e. $u, v \in V$ with $1 \leq d(u, v) \leq r$, shall be called *r-neighbours*. For any colouring $c : E \to \{1, 2, \dots, k\}$, the weighted degree $w_c(v)$ (defined in (1)) shall also be referred to as the *weight* of v or simply the *sum* at v. If $w(u) = w(v)$ for distinct vertices $u, v \in V$, we say that they are *in conflict*, otherwise we call them *sum-distinguished* or simply *distinguished*. The least k such that there is a colouring $c : E \to \{1, 2, \dots, k\}$ without a conflict between any pair of r-neighbours in the graph G shall be called the *r-distant irregularity strength* of G and denoted by $s_r(G)$ (observe that this notion is consistent with the use of $s_1(G)$ with reference to 1–2–3 Conjecture above, while it is also justified to set $s_\infty(G) = s(G)$ in this context). Note that analogously as above, $s_r(G)$ is well defined iff G has no isolated edges. In [24] the following upper bound was provided for this graph invariant.

Theorem 1. *Let G be a graph without isolated edges, and with maximum degree $\Delta \geq 2$, and let $r \geq 1$ be an integer. Then,*

$$s_r(G) \leq 6\Delta^{r-1}.$$

See also [24] for a discussion justifying the fact that the general upper bound from the theorem above cannot be smaller than Δ^{r-1}. In this paper we essentially improve the inequality from Theorem 1 to $s_r(G) \leq (4 + o(1))\Delta^{r-1}$, but for technical reasons we had to exclude from our result graphs with small minimum degrees, i.e., smaller than a value given by a certain poly-logarithmic function in Δ, see Theorem 5 below. Our approach shall be based on the probabilistic method, first applied to design a special ordering of the vertices of a graph, and then to provide an enhancement of an algorithm whose different variants were used e.g. in [14,15,18,24], developed along the specified order. In the next section we recall several useful tools of the probabilistic method. Then we formulate our main result, and provide its proof in Section 4. The last section contains a few related comments.

2. Tools

We shall use a few tools of the probabilistic method listed in details below. In particular, the Lovász Local Lemma, see e.g. [4], combined with the Chernoff Bound, see e.g. [13] (Th. 2.1, page 26) and Talagrand's Inequality, see e.g. [19].

Theorem 2 (*The Local Lemma*). *Let $A_1, A_2, \dots, A_n$ be events in an arbitrary probability space. Suppose that each event A_i is mutually independent of a set of all the other events A_j but at most D, and that $\mathbf{Pr}(A_i) \leq p$ for all $1 \leq i \leq n$. If*

$$ep(D + 1) \leq 1,$$

then $\mathbf{Pr}\left(\bigcap_{i=1}^{n} \overline{A_i}\right) > 0$.

Theorem 3 (*Chernoff Bound*). *For any $0 \leq t \leq np$,*

$$\mathbf{Pr}(\mathrm{BIN}(n, p) > np + t) < e^{-\frac{t^2}{3np}} \text{ and } \mathbf{Pr}(\mathrm{BIN}(n, p) < np - t) < e^{-\frac{t^2}{2np}} \leq e^{-\frac{t^2}{3np}}$$

where $\mathrm{BIN}(n, p)$ is the sum of n independent Bernoulli variables, each equal to 1 with probability p and 0 otherwise.

Theorem 4 (*Talagrand's Inequality*). *Let X be a non-negative random variable, not identically 0, which is determined by l independent trials $T_1, \dots, T_l$, and satisfying the following for some $c, k > 0$:*

1. *changing the outcome of any one trial can affect X by at most c, and*
2. *for any s, if $X \geq s$ then there is a set of at most ks trials whose outcomes certify that $X \geq s$,*

then for any $0 \leq t \leq \mathbf{E}(X)$,

$$\mathbf{Pr}\left(|X - \mathbf{E}(X)| > t + 60c\sqrt{k\mathbf{E}(X)}\right) \leq 4e^{-\frac{t^2}{8c^2k\mathbf{E}(X)}}.$$

Note that e.g. knowing only an upper bound $\mathbf{E}(X) \leq h$ (instead of the exact value of $\mathbf{E}(X)$) we may still use Talagrand's Inequality in order to upper-bound the probability that X is large. It is sufficient to apply Theorem 4 above to the variable $Y = X + h - \mathbf{E}(X)$, with $\mathbf{E}(Y) = h$ to obtain the following provided that the assumptions of Theorem 4 hold for X (and $t \leq h$):

$$\mathbf{Pr}(X > h + t + 60c\sqrt{kh}) \leq \mathbf{Pr}(Y > h + t + 60c\sqrt{kh}) \leq 4e^{-\frac{t^2}{8c^2kh}}.$$