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On extremal cacti with respect to the revised Szeged index ☆

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ABSTRACT

The revised Szeged index of a graph G is defined as $Sz^*(G) = \sum_{e=uv\in E}(n_u(e)+\frac{n_0(e)}{2})(n_v(e)+\frac{n_0(e)}{2})$, where $n_u(e)$ and $n_v(e)$ are, respectively, the number of vertices of G lying closer to vertex u than to vertex v and the number of vertices of G lying closer to vertex v than to vertex u, and $n_0(e)$ is the number of vertices equidistant to u and v. A cactus is a graph in which any two cycles have at most one common vertex. Let $\mathcal{C}(n, k)$ denote the class of all cacti with n vertices and k cycles. In this paper, sharp lower bound on revised Szeged index of graph G in $\mathcal{C}(n, k)$ is established and the corresponding extremal graph is determined. Furthermore, the graph G in $\mathcal{C}(n, k)$ with the second minimal revised Szeged index is identified as well.

1. Introduction

In this paper we are concerned with simple finite graphs. Undefined notation and terminology can be found in [2]. Let G be a connected graph with vertex set $V(G)$ and edge set $E(G)$. For $v \in V(G)$, let $N_G(v)$ (or $N(v)$ for short) denote the set of all neighbors of v in G and $d_G(v) = |N_G(v)|$, the degree of v in G. Call u a *pendent vertex* of G, if $d_G(u) = 1$ and call uv a *pendent edge* of G, if $d_G(u) = 1$ or $d_G(v) = 1$. Denote by P_n, S_n, C_n and K_n the path, star, cycle and complete graph on n vertices, respectively. For $u, v \in V(G)$, let $d_G(u, v)$ denote the distance between u and v in G. The Wiener index of G [30] is defined as

$$W(G) = \sum_{\{u,v\}\subseteq V(G)} d_G(u, v).$$

This topological index has been extensively studied in the mathematical literature, many papers have contributed to the Wiener index and a number of studies mainly focused on determining the lower and upper bounds on Wiener index; see, e.g., [6,8,9,11,12,18,20,21,26,28,34].

Let $e = uv$ be an edge of G, and define three sets as follows:

$$N_u(e) = \{w \in V : d(u, w) < d(v, w)\}, \quad N_v(e) = \{w \in V : d(v, w) < d(u, w)\},$$
$$N_0(e) = \{w \in V : d(u, w) = d(v, w)\}.$$

Thus, $[N_u(e), N_v(e), N_0(e)]$ is a partition of $V(G)$ with respect to e. The number of vertices of $N_u(e)$, $N_v(e)$ and $N_0(e)$ are denoted by $n_u(e)$, $n_v(e)$ and $n_0(e)$, respectively. If G is a tree, then the formula $W(G) = \sum_{e=uv\in E(G)} n_u(e)n_v(e)$ gives a long time known property of the Wiener index [12,30].

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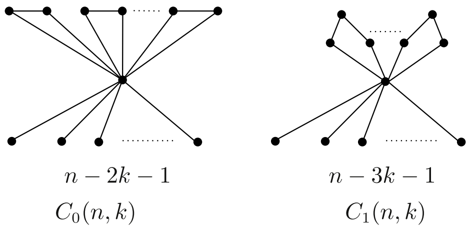

Fig. 1. The graphs $C_0(n, k)$ and $C_1(n, k)$.

Using the above formula, Gutman [10] introduced a graph invariant named the Szeged index as an extension of the Wiener index and defined it by

$$Sz(G) = \sum_{e=uv\in E(G)} n_u(e)n_v(e).$$

Randić [26] observed that the Szeged index does not take into account the contributions of the vertices at equal distances from the endpoints of an edge, and so he conceived a modified version of the Szeged index which is named the revised Szeged index. The revised Szeged index of a connected graph G is defined as

$$Sz^*(G) = \sum_{e=uv\in E(G)} \left(n_u(e) + \frac{n_0(e)}{2}\right)\left(n_v(e) + \frac{n_0(e)}{2}\right).$$

There are several results on the difference (resp. quotient) between the Szeged index (resp. revised Szeged index) and Wiener index; see, e.g., [3,5,7,8,14–16,19,22–24,32,33]. For some properties and applications of the Szeged index and the revised Szeged index one may be referred to [1,3,5,13,17,24,25,27,29,31].

Aouchiche and Hansen [1] showed that for a connected graph G of order n and size m, an upper bound of the revised Szeged index of G is $\frac{n^2m}{4}$. Xing and Zhou [31] determined the unicyclic graphs of order n with the smallest and the largest revised Szeged indices for $n \geq 5$. Li and Liu [17] characterized the bicyclic graphs of order n with the largest revised Szeged indices for $n \geq 6$. Chen, Li and Liu [4] established an upper bound of the revised Szeged index for a connected tricyclic graph, and also characterized those graphs that achieve the upper bound.

A cactus is a graph in which any block is either a cut edge or a cycle. It is also a graph in which any two cycles have at most one common vertex. A cycle in a cactus is called *end-block* if all but one vertex of this cycle have degree 2. If all the cycles in a cactus have exactly one common vertex, then they form a *bundle*. Let $\mathcal{C}(n, k)$ be the class of all cacti of order n with k cycles. In this paper, sharp lower bound on revised Szeged index of graph G in $\mathcal{C}(n, k)$ is established and the corresponding extremal graph is determined. Furthermore, when $n \geq 13$ and $n > 3k + 2$, the graph G in $\mathcal{C}(n, k)$ with the second minimal revised Szeged index is identified as well. Let $C_0(n, k) \in \mathcal{C}(n, k)$ be the bundle of k triangles with $n - 2k - 1$ pendent vertices attached to the common vertex and let $C_1(n, k) \in \mathcal{C}(n, k)$ be the bundle of k quadrangles with $n - 3k - 1$ pendent vertices attached to the common vertex; see Fig. 1.

2. Preliminaries

In this section, we give some preliminary results which will be used in the subsequent sections. Using the fact that $n_u(e) + n_v(e) + n_0(e) = n$, we have that

$$\begin{aligned} Sz^*(G) &= \sum_{e=uv\in E(G)} \left(n_u(e) + \frac{n_0(e)}{2}\right)\left(n_v(e) + \frac{n_0(e)}{2}\right) \\ &= \sum_{e=uv\in E(G)} \left(\frac{n + n_u(e) - n_v(e)}{2}\right)\left(\frac{n + n_v(e) - n_u(e)}{2}\right) \\ &= \sum_{e=uv\in E(G)} \frac{n^2 - (n_u(e) - n_v(e))^2}{4} \\ &= \frac{n^2m}{4} - \frac{1}{4}\sum_{e=uv\in E(G)} (n_u(e) - n_v(e))^2. \end{aligned}$$

Moreover, if $G \in \mathcal{C}(n, k)$, then $m = n + k - 1$. Consequently,

$$Sz^*(G) = \frac{n^2(n + k - 1)}{4} - \frac{1}{4}\sum_{e=uv\in E(G)} (n_u(e) - n_v(e))^2. \tag{2.1}$$