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On the strong Roman domination number of graphs

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ABSTRACT

Based on the history that the Emperor Constantine decreed that any undefended place (with no legions) of the Roman Empire must be protected by a "stronger" neighbor place (having two legions), a graph theoretical model called Roman domination in graphs was described. A Roman dominating function for a graph $G = (V, E)$, is a function $f : V \to \{0, 1, 2\}$ such that every vertex v with $f(v) = 0$ has at least a neighbor w in G for which $f(w) = 2$. The Roman domination number of a graph is the minimum weight, $\sum_{v\in V} f(v)$, of a Roman dominating function. In this paper we initiate the study of a new parameter related to Roman domination, which we call strong Roman domination number and denote it by $\gamma_{StR}(G)$. We approach the problem of a Roman domination-type defensive strategy under multiple simultaneous attacks and begin with the study of several mathematical properties of this invariant. In particular, we first show that the decision problem regarding the computation of the strong Roman domination number is NP-complete, even when restricted to bipartite graphs. We obtain several bounds on such a parameter and give some realizability results for it. Moreover, we prove that for any tree T of order $n \geq 3$, $\gamma_{StR}(T) \leq 6n/7$ and characterize all extremal trees.

1. Introduction

The Roman domination in graphs was introduced by Cockayne et al. [10], according to some connections with historical problems of defending the Roman Empire described in [22,25]. A *Roman dominating function* (RDF for short) on a graph $G = (V, E)$ is defined as a function $f : V \longrightarrow \{0, 1, 2\}$ satisfying the condition that every vertex v for which $f(v) = 0$ is adjacent to at least one vertex u for which $f(u) = 2$. The *weight* of an RDF f is the value $\omega(f) = \sum_{v\in V} f(v)$. The *Roman domination number* of a graph G, denoted by $\gamma_R(G)$, equals the minimum weight of an RDF on G. A $\gamma_R(G)$-*function* is a Roman dominating function of G with weight $\gamma_R(G)$. After this seminal work [10], several investigations have been focused into obtaining properties of this invariant [12,13,16,17,26].

On the other hand, in order to generalize or improve some particular property of the Roman domination in its standard presentation, some variants of Roman domination have been introduced and studied. Those variants are frequently related

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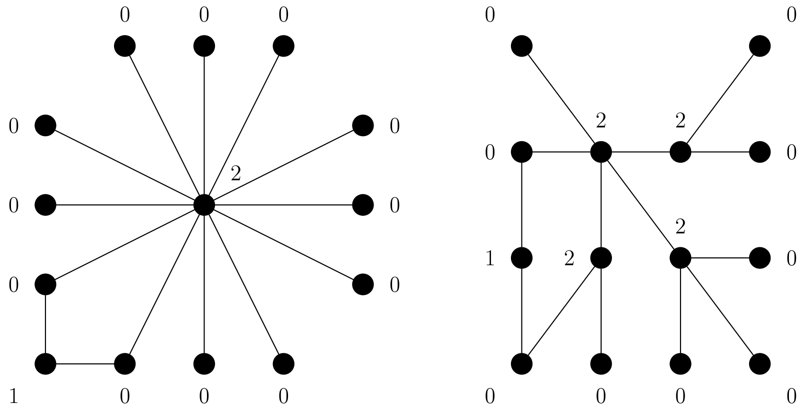

Fig. 1. Two Roman dominating functions.

to modifying the way in which the vertices are dominated, or to adding an extra property to the Roman domination property itself. For instance, we mention variants like the following ones: independent Roman domination [4,8], edge Roman domination [23], weak Roman domination [9]; [18], total Roman domination[1] [21], signed Roman domination [2,24], signed Roman edge domination [1], distance Roman domination [5] and Roman k-domination [15,19], among others. On the other hand, an interesting version regarding the defense of the "Roman Empire" against multiple sequential attacks (not simultaneously) was described in [17]. For other studies on protecting graphs against sequential attacks we suggest the recent survey [20]. In this article we propose a new version of Roman domination in which we also deal with multiple attacks, but simultaneously in our case.

To begin with our work, we first introduce the terminology and notation we shall use throughout the exposition. Unless otherwise stated, other notation and terminology not explicitly given here can be found in [7]. Let G be a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The *order* $|V|$ of G is denoted by $n = n(G)$ and the *size* $|E|$ of G is denoted by $m = m(G)$. By $u \sim v$ we mean that u, v are adjacent, *i.e.*, $uv \in E$. For a non-empty set $X \subseteq V$, and a vertex $v \in V$, $N_X(v)$ denotes the set of neighbors v has in X, or equivalently, $N_X(v) = \{u \in X : u \sim v\}$. In the case $X = V$, we use only $N(v)$, instead of $N_V(v)$, which is also called the *open neighborhood* of the vertex $v \in V$. The *closed neighborhood* of a vertex $v \in V$ is $N[v] = N(v) \cup \{v\}$. For any vertex v, the cardinality of $N(v)$ is the *degree* of v in G, denoted by $\deg_G(v)$ (or just $\deg(v)$ if confusion is unlikely). The *minimum* and the *maximum degree* of a graph G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. The *open neighborhood* of a set $S \subseteq V$ is the set $N(S) = \cup_{v \in S} N(v)$, and the *closed neighborhood* of S is the set $N[S] = N(S) \cup S$. A *universal vertex* of G is a vertex which is adjacent to every other vertex of G.

A uv-*path* in G, joining the (end) vertices $u, v \in V$, is a finite alternating sequence: $u_0 = u, e_1, u_1, e_2, \dots, u_{k-1}, e_k, u_k = v$ of different vertices and edges, beginning with the vertex u and ending with the vertex v, so that $e_i = u_{i-1}u_i$ for all $i = 1, 2, \dots, k$. The number of edges in a path is called the *length* of the path. The length of a shortest uv-path is the *distance between the vertices* u and v, and it is denoted by $d(u, v)$. The maximum among all the distances between two vertices in a graph G is denoted by $Diam(G)$, the *diameter* of G. A *cycle* is a uu-path. The *girth* of a graph G, denoted by $g(G)$, is the length of its shortest cycle. The girth of a graph with no cycle is defined to be ∞.

The set of vertices $D \subset V$ is a *dominating set* if every vertex v not in D is adjacent to at least one vertex in D. The minimum cardinality of any dominating set of G is the *domination number* of G and is denoted by $\gamma(G)$. A dominating set D in G with $|D| = \gamma(G)$ is called a $\gamma(G)$-set. Notice that a graph having a universal vertex has domination number equal to one.

Let f be a Roman dominating function for G and let $V(G) = A_0 \cup A_1 \cup A_2$ be the sets of vertices of G induced by f, where $A_i = \{v \in V : f(v) = i\}$, for all $i \in \{0, 1, 2\}$. It is clear that for any Roman dominating function f of a graph G, we have that $f(V) = \sum_{u \in V} f(u) = 2|A_2| + |A_1|$. A Roman dominating function f can be represented by the ordered partition (A_0, A_1, A_2) of $V(G)$. It is proved that for any graph G, $\gamma(G) \leq \gamma_R(G) \leq 2\gamma(G)$ [10]. Note that if $C_1, C_2, \dots, C_t$ are the components of G, then $\gamma_R(G) = \sum_{i=1}^{t} \gamma_R(C_i)$. Therefore, from now on we will only consider connected graphs, unless it would be necessary to satisfy some specific condition.

The defensive strategy of Roman domination is based on the fact that every place in which there is established a Roman legion (a label 1 in the Roman dominating function) is able to protect itself under external attacks; and that every unsecured place (a label 0) must have at least a stronger neighbor (a label 2). In that way, if an unsecured place (a label 0) is attacked, then a stronger neighbor could send one of its two legions in order to defend the weak neighbor vertex (label 0) from the attack. Two examples of Roman dominating functions are depicted in Fig. 1.

Although these two functions (Fig. 1) satisfy the conditions to be Roman dominating functions, they correspond to two very different real situations. The unique strong place (2) in the left hand side graph must defend up to 12 weak places from possible external attacks. However, in the right hand side graph, the task of defending the unsecured places is divided

[1] The concept of total Roman domination was introduced in [21] albeit in a more general setting. Its specific definition has appeared in [3].

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