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Pseudoachromatic and connected-pseudoachromatic indices of the complete graph ☆

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ABSTRACT

A *complete k-coloring* of a graph G is a (not necessarily proper) k-coloring of the vertices of G, such that each pair of different colors appears in an edge. A complete k-coloring is also called *connected*, if each color class induces a connected subgraph of G. The *pseudoachromatic index* of a graph G, denoted by $\psi'(G)$, is the largest k for which the line graph of G has a complete k-coloring. Analogously the *connected-pseudoachromatic index* of G, denoted by $\psi'_c(G)$, is the largest k for which the line graph of G has a connected and complete k-coloring.

In this paper we study these two parameters for the complete graph K_n. Our main contribution is to improve the linear lower bound for the connected pseudoachromatic index given by Abrams and Berman (2014) and provide an upper bound. These two bounds prove that for any integer $n \geq 8$ the order of $\psi'_c(K_n)$ is $n^{3/2}$.

Related to the pseudoachromatic index we prove that for q a power of 2 and $n = q^2 + q + 1$, $\psi'(K_n)$ is at least $q^3 + 2q - 3$ which improves the bound $q^3 + q$ given by Araujo et al. (2011).

1. Introduction

Let $G = (V(G), E(G))$ be a finite simple graph. A *complete* k-coloring of G is an assignment $\varsigma : V(G) \to [k]$ (where $[k] := \{1, \ldots, k\}$), such that for each pair of different colors $i, j \in [k]$ there exists an edge $xy \in E(G)$ where $x \in \varsigma^{-1}(i)$ and $y \in \varsigma^{-1}(j)$. The *pseudoachromatic number* $\psi(G)$ of G is the largest k for which there exists a complete k-coloring of G [12]. Some interesting results on the pseudoachromatic number and the closely related notion of the achromatic number (the maximum in proper and complete colorings) can be found in [6,8,14].

The *connected-pseudoachromatic number* $\psi_c(G)$ of a connected graph G is the largest k for which there exists a connected and complete k-coloring of G, i.e., a complete k-coloring in which each color class induces a connected subgraph. The connected pseudoachromatic number $\psi_c(G)$ of a graph G with components $G_1, G_2, \ldots, G_t$ is the largest number between $\psi_c(G_1), \psi_c(G_2), \ldots, \psi_c(G_t)$. Clearly,

$$\psi_c(G) \leq \psi(G).$$

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Table 1
Exact values for $\psi'_c(K_n)$, $2 \le n \le 7$.

n	2	3	4	5	6	7
$\psi'_c(K_n)$	1	3	4	6	7	10

Table 2
Exact values for $\psi'(K_n)$, $2 \le n \le 13$.

n	2	3	4	5	6	7	8	9	10	11	12	13
$\psi'(K_n)$	1	3	4	7	8	11	14	18	22	27	32	39

The previous definition is equivalent to saying that $\psi_c(G)$ is the size of the largest complete graph minor of G. This value is also called the *Hadwiger number* of G. In 1958 Hadwiger conjectured (see [13]) that any graph G, $\chi(G) \le \psi_c(G)$, where $\chi(G)$ denotes the chromatic number of G as usual. Aside from its own importance the Hadwiger Conjecture is another motivation to study the previous parameters given the pseudoachromatic number bounding the Hadwiger number.

The connected-pseudoachromatic and the pseudoachromatic numbers of the line graph $L(G)$ of a graph G are also known as the *connected-pseudoachromatic index* and the *pseudoachromatic index* of G denoted as $\psi'_c(G)$ and $\psi'(G)$ respectively. In this paper we study these two parameters for the complete graph K_n. Note that any connected and complete k-coloring of $L(K_n)$ is an edge coloring of K_n in which each edge color class induces a connected subgraph and each pair of edge color classes shares at least one vertex, therefore, we will make use of this point of view.

Note that the colorations that induce the lower bound are given using the structure of projectives planes, it is a common technique used to obtain results for the pseudoachromatic and achromatic indices of the complete graph K_n, for instance see [1,2,8,4,9,16].

Before continuing we give some known results on $\psi'_c(K_n)$ and $\psi'(K_n)$ related to our contributions. In the case of exact values of $\psi'_c(K_n)$ only the following small values of Table 1 are known. Table 1 appears in [1], which also proves that $\psi'_c(K_{5a+b+1})$ is at least $9a+b$.

However, in Table 2 (which appears also in [2]) the first pseudoachromatic index values of the complete graph are listed.

The pseudoachromatic index for a few classes of complete graphs has been found so far, namely, Bouchet proved in [9] that for $n = q^2+q+1$ and q an odd integer $\alpha'(n) = qn$, if and only if the projective plane of order q exists. Consequently, it is known $\psi'(K_{q^2+q+1})$ for any odd prime power q (see [4,9]), however, when q is a power of 2 the pseudoachromatic index is bounding by $\psi'(K_{q^2+q+1}) \ge q^3+q$ (see [4]) and attains exact values for $K_{q^2+2q+1-a}$ when $a \in \{-1, 0, \dots, \left\lceil \frac{1+\sqrt{4q+9}}{2} \right\rceil - 1\}$ (see [3,2,4]). Finally, it is proved that $\psi'(K_n)$ grows asymptotically like $n^{3/2}$ (see [16]).

This paper is organized as follows: In the first section we give some definitions and lemmas related to projective planes and colorings, which will be used in the second and third sections in order to give lower bounds for the pseudoachromatic connected and pseudoachromatic indices of complete graphs. In Section 2 we prove that for any integer $n \ge 8$, $\psi'_c(K_n)$ has order $n^{3/2}$. The proof is divided in the following results:

Theorem 1. *If $n \ge 8$ then*

$$\psi'_c(K_n) \le \lfloor \max \{\min\{f_n(x), g_n(x)\} \text{ with } x \in \mathbb{N}\} \rfloor$$

where $f_n(x) = n(n-1)/2x$ and $g_n(x) = (x+1)(n-x-1/2)$.

Theorem 2. *If $n \ge 8$ then*

$$\psi'_c(K_n) \le \frac{1}{\sqrt{2}} n^{\frac{3}{2}} + \Theta(n).$$

Theorem 3. *If q is a prime power and $n = q^2+q+1$ then*

$$\left\lceil \frac{q}{2} \right\rceil n \le \psi'_c(K_n).$$

Theorem 4. *If $n \ge 8$ then*

$$\frac{1}{2} n^{\frac{3}{2}} + \Theta(n) \le \psi'_c(K_n).$$

Comparing the results of Theorems 1 and 2 with the upper bounds for the pseudoachromatic number given in [3,2,4,16] we have that $\psi'_c(K_n) < \psi'(K_n)$ since $\psi'(K_n) = n^{\frac{3}{2}} + \Theta(n)$. Furthermore, Theorems 3 and 4 improve the linear lower bound