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Rerouting shortest paths in planar graphs[☆]

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ABSTRACT

A rerouting sequence is a sequence of shortest st-paths such that consecutive paths differ in one vertex. We study the Shortest Path Rerouting Problem, which asks, given two shortest st-paths P and Q in a graph G, whether a rerouting sequence exists from P to Q. This problem is PSPACE-hard in general, but we show that it can be solved in polynomial time if G is planar. In addition, it can be solved in polynomial time when every vertex has at most two neighbors closer to s and at most two neighbors closer to t. To this end, we introduce a dynamic programming method for reconfiguration problems, based on using small encodings of reconfiguration graphs.

1. Introduction

In this paper, we study the *Shortest Path Rerouting (SPR)* Problem, as introduced by Kamiński et al. [21]. An instance of this problem consists of a graph G with unit edge lengths, two vertices $s, t \in V(G)$, and two shortest st-paths P and Q. Shortest st-paths are *adjacent* if they differ in one vertex. The question is whether there exists a *rerouting sequence* from P to Q, which is a sequence of shortest st-paths $Q_0, \dots, Q_k$ with $Q_0 = P$, $Q_k = Q$, such that consecutive paths are adjacent. This question may arise for instance when a commodity is routed in a network along a shortest path, and a different shortest path route is desired. However, changing the path can only be done by exchanging one node at a time, and transfer should not be interrupted [21]. A different setting where this problem may occur is if a shortest path changes randomly over time, and one wishes to know which paths are reachable from a given starting path.

On the negative side, this problem is known to be PSPACE-complete [4]. Furthermore, Kamiński et al. [21] describe instances where the minimum length of a rerouting sequence is exponential in $n = |V(G)|$. In addition they show that it is strongly *NP*-hard to decide whether a rerouting sequence of length at most k exists [21]. On the positive side, in [4] it is shown that SPR can be solved in polynomial time in the case where G is claw-free or chordal. In these cases, the related problem of deciding whether there is a rerouting sequence between *every* pair of shortest st-paths is also shown to be solvable in polynomial time, and in the chordal case, a shortest rerouting sequence can be found efficiently.

In this paper, we study the SPR Problem for planar graphs, and prove that this case can also be solved in polynomial time. Questions related to (rerouting) shortest paths occur often in networks, and these are often planar in practice, so this is a relevant graph class for this problem.

Similar questions about the reachability of solutions can be asked for many different combinatorial problems. This only requires defining a (symmetric) adjacency relation between feasible solutions. Ito et al. [18] called such problems

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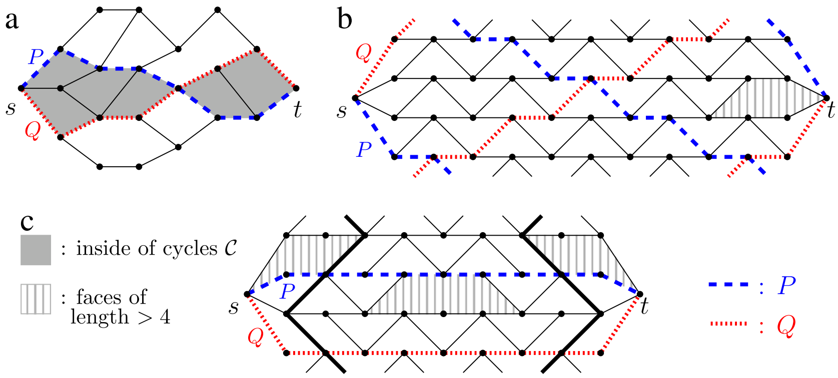

Fig. 1. Three examples of planar graphs where a rerouting sequence from P to Q exists. In all of our figures, half edges leaving the top of the figure continue on the bottom.

reconfiguration problems, and initiated a systematic study of them. To be precise, for a reconfiguration problem, it is necessary that both adjacency between solutions and the property of being a solution can be tested in polynomial time [18]. Such reconfiguration problems have been studied often in recent literature, for instance based on vertex colorings [7–10,15,20,25], independent sets [5,6,16,18,24], dominating sets [14,24] satisfying assignments for boolean formulas [13,27], matchings [18], and more [12,18,19]. Usually, the most natural adjacency relation between solutions is considered, e.g. two vertex colorings are considered adjacent in [7–10,20,25] if they differ in one vertex.

One of the motivations for researching reconfiguration problems is to study the structure of the solution space of well-studied combinatorial problems, which can explain the performance of various heuristics [9,13]. In addition, similar problems have also occurred in practical applications such as stacking problems in storage spaces [23] and train switchyards (see [22] and references therein). See the recent survey by Van den Heuvel [17] for more information on reconfiguration problems and motivations.

The main question that has been studied for reconfiguration problems is the complexity of deciding whether there exists a reconfiguration sequence between a given pair of solutions, which we call the *reachability problem*. For most of the aforementioned reconfiguration concepts the reachability problem turned out to be PSPACE-complete in general. Alternatively, one may ask whether there exist reconfiguration sequences between *all* solutions [4,9,13], or study the question of how much the solution space needs to be increased such that a reconfiguration sequence becomes possible [18,25], e.g. allowing to use more colors in vertex colorings used in a reconfiguration sequence [25]. In addition, one may study the problem of finding shortest reconfiguration sequences [7,20,21,27], or give upper bounds on their length [2,10,13].

Initially, most of the more advanced (computational complexity) results on reconfiguration problems, in particular reachability problems, were negative results, such as the early PSPACE-hardness results in [16,13]. In recent years, some more advanced positive results (polynomial time algorithms) have been obtained for special instances of PSPACE-hard reconfiguration problems [4–6,10,14]. In addition, many fixed parameter tractable algorithms have been obtained for various parameterizations of reconfiguration problems (see e.g. [7,20,24,28,27]), after the parameterized viewpoint on reconfiguration problems was introduced in [26]. Nevertheless, there is still a lack of general algorithmic techniques for reconfiguration problems. Introducing such techniques, and further showing that advanced positive results are possible for reconfiguration problems, are important motivations for the research presented in this paper.

We now give an outline of our results, an informal introduction to the new techniques and ideas, and sketch some obstacles for SPR in planar graphs. Detailed definitions will be given afterwards in Section 2. Firstly, in polynomial time we can delete all vertices and edges of G that do not lie on any shortest st-path, without affecting the answer. So we may assume that all vertices and edges of G lie on shortest st-paths. The problem is then straightforward in the case where the given graph G is st-planar, i.e. has a (planar) embedding where the end vertices s and t are incident with the same face, say the infinite face. The symmetric difference $\mathcal{C} := E(P) \Delta E(Q)$ of the edge sets of the two given shortest st-paths P and Q gives a set of cycles, with disjoint insides (an embedded cycle divides the plane into two regions; the finite region is called the *inside*). One can easily verify that a rerouting sequence from P to Q exists if and only if all faces inside these cycles $\mathcal{C}$ have length 4. See Fig. 1(a) for an example. If G is not st-planar, there are still cases where such a topological viewpoint works well: Fig. 1(b) shows an example where every rerouting sequence $Q_0, \ldots, Q_k$ has the property that for any pair of consecutive shortest paths Q_i and Q_{i+1}, the symmetric difference of the edge sets $E(Q_i) \Delta E(Q_{i+1})$ gives a facial 4-cycle. (For readability, some edges are shown as pairs of half edges in our figures; edges leaving the top of the figure continue on the bottom.) Sloppily speaking, rerouting sequences of this type are called *topological*. Even though it is not clear in advance which region of the plane should have only faces of length 4, using some topological intuition it can be verified that also in this example a rerouting sequence from P to Q exists (the unique face of length greater than 4 is shaded). However, not all rerouting sequences in planar graphs are topological. Fig. 1(c) shows an example where there exists no topological rerouting sequence from P to Q (consider the shaded faces). The method given in Section 6 will show that nevertheless,

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