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Colorful edge decomposition of graphs: Some polynomial cases

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ABSTRACT

Consider a graph G and a labeling of its edges with r labels. Every vertex $v \in V(G)$ is associated with a palette of incident labels together with their frequencies, which sum up to the degree of v. We say that two vertices have distinct palettes if they differ in the frequency of at least one label, otherwise they have the same palette. For an integer $r > 0$, a colorful r-edge decomposition of a graph G is a labeling $\ell : E(G) \to \{1, \dots, r\}$ such that for any vertex $v \in V(G)$, the edges incident with the vertex v have at least $\min\{r, d(v)\}$ different labels and for every two adjacent vertices v and u with $d(v) = d(u)$, they have the same palette.

In this work, we investigate some properties of this edge decomposition. We prove that for each t, every tree has a colorful t-edge decomposition and that decomposition can be found in polynomial time. On the negative side, we prove that for a given graph G with $2 \leq \delta(G) \leq \Delta(G) \leq 3$ determining whether G has a colorful 2-edge decomposition is **NP**-complete. Among other results, we show that for a given bipartite graph G with degree set $\{a_1, a_2, \dots, a_c\}$ where c is a constant number, if for each i, $1 \leq i \leq c$, the induced subgraph on the set of vertices of degree a_i has d connected components, where d is a constant number, then there is a polynomial time algorithm to decide whether G has a colorful 2-edge decomposition. Also, we prove that every r-regular graph G with at most one cut edge has a colorful 2-edge decomposition.

1. Introduction

A *simple graph* is a graph in which both multiple edges and loops are disallowed. Throughout this paper, all graphs are finite, undirected and simple. We follow [24] for terminology and notations which are not defined here. We denote the vertex set and the edge set of a graph G by $V(G)$ and $E(G)$, respectively. Also, for every $v \in V(G)$, $d(v)$ and $N(v)$ denote the degree of v and the neighbor set of v, respectively. We denote the maximum degree and the minimum degree of G by $\Delta(G)$ and $\delta(G)$, respectively. A *cut edge* is an edge whose removal increases the number of connected components. Also, a graph is *k-edge connected* if the minimum number of edges whose removal disconnects the graph is at least k. A subgraph F of a graph G is called a *factor* of G if F is a spanning subgraph of G. A k-regular factor is called a *k-factor*. The *degree set* D of a graph G is the set of distinct degrees of the vertices of G.

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1.1. Statement of the problem and related work

Consider a graph G and an edge labeling $\ell : E(G) \to \{1, 2, \ldots, r\}$. Such a labeling is called *neighbor distinguishing* if for every edge $uv \in E(G)$, the multiset of labels incident with u is distinct from the multiset of labels incident with v. In other words, if for every vertex v, we set $\overline{\ell}(v) = (a_1, \ldots, a_r)$, where $a_i = |\{w : wv \in E, \ell(wv) = i\}|$ for $i = 1, \ldots, r$, then the labeling ℓ is neighbor distinguishing if $\overline{\ell}(v) \neq \overline{\ell}(u)$ for each edge uv of G. It was shown that every graph G has a neighbor distinguishing labeling with four labels [1]. Also, if G has the minimum degree at least one thousand, then there is a partition of $E(G)$ into three sets such that the corresponding multisets yield a neighbor distinguishing labeling [1]. Furthermore, it was proved that determining whether a given 3-regular graph G has a neighbor distinguishing labeling with two labels, is **NP**-complete [6].

Consider a graph G and a labeling of its edges with r labels. Then every vertex $v \in V(G)$ is associated with a palette of incident labels together with their frequencies, which sum up to the degree of v. We say that two vertices have distinct palettes if they differ in the frequency of at least one label, otherwise they have the same palette (throughout the paper, when we say the palette of a vertex v we mean the multiset of colors of edges incident to v).

For an integer $r > 0$, a colorful r-edge decomposition of a graph G is a labeling $\ell : E(G) \to \{1, \ldots, r\}$ such that for any vertex $v \in V(G)$, the edges incident with the vertex v have at least $\min\{r, d(v)\}$ different labels and for every two adjacent vertices v and u with $d(v) = d(u)$, they have the same palette.

Note that another type of palette was also investigated by Horňák et al. in [11]. Consider a graph G and a proper edge coloring $f : E(G) \to \{1, 2, \ldots, r\}$ for it (i.e., any two adjacent edges get distinct colors). Hornak et al. defined the palette of a vertex $v \in V(G)$ with respect to f as the **set** of colors of edges incident to v. They worked on the minimum number of palettes taken over all possible proper edge colorings of a graph. For more information about this coloring, see [11,14].

1.2. Motivation

1.2.1. Regular graphs

Let G be an r-regular graph. A k-factorization of a graph G is a partition of $E(G)$ into subsets inducing (edge disjoint) k-factors. A colorful t-edge decomposition is a generalization of a k-factorization in regular graphs. In other words, a colorful t-edge decomposition of G is a partitioning $E_1, E_2, \ldots, E_t$ of edges of G, such that for every $i = 1, \ldots, t$, part E_i is an r_i-factor of G and $\sum_i r_i = r$.

1.2.2. General graphs

For a family $\mathcal{Q}$ of graphs, a $\mathcal{Q}$-edge decomposition of a graph G is a partition of the edge set of G into subgraphs isomorphic to members of $\mathcal{Q}$. Problems of $\mathcal{Q}$-edge decomposition of graphs have received considerable attention, for example, Holyer proved that it is **NP**-hard to edge-partition a graph into the minimum number of complete subgraphs [10]. For more examples, see [3,7,13,15–17,21,22]. Our investigations are motivated by the following decomposition problem:

Subgraph Decomposition Problem (SDP):

Instance: A graph G.

Question: Can $E(G)$ be partitioned into subsets inducing subgraphs of G that are all similar to G?

We can determine whether a graph G is similar to its subgraph H through different parameters (for example we can say that two graphs are similar if they have the same eigenvalues or they have equal diameter). Let G be a graph. The graph H is called a *semi-subgraph* of a graph G, if H is a subgraph of G, and for every two adjacent vertices v and u in G, with $d_G(v) = d_G(u)$, they have a same degree in H. A colorful r-edge decomposition for a graph G is a partition of $E(G)$ into subsets $E_1, E_2, \ldots, E_r$ such that for each $i = 1, \ldots, r$, the induced subgraph on the edge set E_i is a semi-subgraph of the graph G and every vertex v is in at least $\min\{d(v), r\}$ induced subgraphs.

1.3. Summary of results

1.3.1. Trees

In this section, we show that for each integer k, every tree of order n has a colorful k-edge decomposition and this decomposition can be found in polynomial time. Our polynomial time algorithm needs $\mathcal{O}(n^2)$ space. Finding an algorithm that runs in linear time with $\mathcal{O}(1)$ space can be interesting.

Theorem 1. *For each integer k, every tree has a colorful k-edge decomposition and this decomposition can be found in polynomial time.*

1.3.2. Regular and semi-regular graphs

The edge chromatic number of a graph denoted by $\chi'(G)$ is the minimum size of a partition of the edge set into 1-regular subgraphs. By Vizing's Theorem [23], the edge chromatic number of a graph G is equal to either $\Delta(G)$ or $\Delta(G) + 1$. It was

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