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# Two more characterizations of König–Egerváry graphs

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ABSTRACT

Let $G$ be a simple graph with vertex set $V(G)$. A set $S \subseteq V(G)$ is *independent* if no two vertices from $S$ are adjacent. The graph $G$ is known to be König–Egerváry if $\alpha(G) + \mu(G) = |V(G)|$, where $\alpha(G)$ denotes the size of a maximum independent set and $\mu(G)$ is the cardinality of a maximum matching. A nonempty collection $\Gamma$ of maximum independent sets is *König–Egerváry* if $\left|\bigcup \Gamma\right| + \left|\bigcap \Gamma\right| = 2\alpha(G)$ (Jarden et al., 2015).

In this paper, we prove that $G$ is a König–Egerváry graph if and only if for every two maximum independent sets $S_1, S_2$ of $G$, there is a matching from $V(G) - S_1 \cup S_2$ into $S_1 \cap S_2$. Moreover, the same is true, when instead of two sets $S_1$ and $S_2$ we consider an arbitrary König–Egerváry collection.



## 1. Introduction

Throughout this paper $G$ is a finite simple graph with vertex set $V(G)$ and edge set $E(G)$. If $X \subseteq V(G)$, then $G[X]$ is the subgraph of $G$ induced by $X$. By $G - W$ we mean either the subgraph $G[V(G) - W]$, if $W \subseteq V(G)$, or the subgraph obtained by deleting $W \subseteq E(G)$. In both cases, we use $G - w$, whenever $W = \{w\}$.

A set $S \subseteq V(G)$ is *independent* if no two vertices from $S$ are adjacent, and by $\mathrm{Ind}(G)$ we mean the family of all the independent sets of $G$. An independent set of maximum size is a *maximum independent set* of $G$, and $\alpha(G) = \max\{|S| : S \in \mathrm{Ind}(G)\}$.

Let $\Omega(G)$ denote the family of all maximum independent sets,

$$\mathrm{core}(G) = \bigcap\{S : S \in \Omega(G)\}[15], \quad \text{and}$$
$$\mathrm{corona}(G) = \bigcup\{S : S \in \Omega(G)\}[2].$$

A *matching* is a set $M$ of pairwise non-incident edges of $G$. *$M$ matches $A$ into $B$* if every vertex of $A$ is matched with some vertex of $B$ by an edge belonging to $M$. A matching of maximum cardinality, denoted $\mu(G)$, is a *maximum matching*. For every matching $M$, we denote the set of all vertices that $M$ saturates by $V(M)$, and by $M(x)$ we denote the vertex $y$ satisfying $xy \in M$.

It is well-known that $\alpha(G) + \mu(G) \leq |V(G)|$ holds for every graph $G$. Recall that if $\alpha(G) + \mu(G) = |V(G)|$, then $G$ is a *König–Egerváry graph* [7,26]. For example, each bipartite graph is a König–Egerváry graph as well. Various characterizations of König–Egerváry graphs can be found in [1,6,7,9,12–14,16–18,20,21,24–26], and some generalizations appear in [3,22].

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A set $A \subseteq V(G)$ is a *clique* in $G$ if $A$ is independent in $\overline{G}$, and $\omega(G) = \alpha(\overline{G})$, where $\overline{G}$ denotes the complement of $G$.

**Lemma 1.1** (*Clique Collection Lemma [8]*)**.** *If $\Lambda$ is a nonempty collection of maximum cliques in G, then*

$$\left|\bigcap \Lambda\right| \geq 2 \cdot \omega(G) - \left|\bigcup \Lambda\right|.$$

Some recent applications of the Clique Collection Lemma may be found in [4,5,11,23]. In [19] we proved that this lemma is a numerical consequence of the following.

**Lemma 1.2** (*Matching Lemma [19]*)**.** *If $A \in \mathrm{Ind}(G)$, $\Gamma \subseteq \Omega(G)$, and $|\Gamma| \geq 1$, then there exists a matching from $A - \bigcap \Gamma$ into $\bigcup \Gamma - A$.*

Since every maximum clique of $G$ is a maximum independent set of $\overline{G}$, the Clique Collection Lemma is equivalent to the fact that the inequality

$$\left|\bigcup \Gamma\right| + \left|\bigcap \Gamma\right| \geq 2\alpha(G)$$

holds for every nonempty family $\Gamma \subseteq \Omega(G)$. The extremal case of this inequality brings about the following definition, which allows us to formulate the main findings of this paper. A nonempty family $\Gamma \subseteq \Omega(G)$ is a *König–Egerváry collection* if

$$\left|\bigcup \Gamma\right| + \left|\bigcap \Gamma\right| = 2\alpha(G) \text{ [10]}.$$

Clearly, each $\Gamma \subseteq \Omega(G)$ with $|\Gamma| \in \{1, 2\}$ is a König–Egerváry collection.

**Theorem 1.3** (*[9,10]*)**.** *If G is a König–Egerváry graph, then every non-empty collection of maximum independent sets is König–Egerváry.*

In particular, taking $\Gamma = \Omega(G)$ in Theorem 1.3, one can obtain the following.

**Corollary 1.4** (*[19]*)**.** *If G is a König–Egerváry graph, then $\Omega(G)$ is a König–Egerváry collection.*

In this paper, we provide two new characterizations of König–Egerváry graphs. One of them is a kind of converse of Corollary 1.4. The other one illustrates the same phenomenon as emphasized in the following.

**Theorem 1.5** (*[18]*)**.** *For a graph G, the following properties are equivalent:*

(i) *G is a König–Egerváry graph;*
(ii) *for every $S \in \Omega(G)$, each maximum matching of G matches $V(G) - S$ into S;*
(iii) *there exists $S \in \Omega(G)$ such that each maximum matching of G matches $V(G) - S$ into S.*

Namely, it is enough to claim something about one maximum independent set in order to be sure that the same is true for all maximum independent sets.

## 2. Results

Let us consider the graph $G$ from Fig. 1, and $\Gamma = \{S_1, S_2, S_3\} \subset \Omega(G)$, where $S_1 = \{a, b, d, u\}$, $S_2 = \{a, b, d, y\}$ and $S_3 = \{a, b, x, v\}$. The subgraph $G\left[\bigcup \Gamma\right]$ is not a König–Egerváry graph, and no perfect matching exists in $G\left[\bigcup \Gamma - \bigcap \Gamma\right]$. Notice that $\Gamma$ is not a König–Egerváry collection, since $\left|\bigcup \Gamma\right| + \left|\bigcap \Gamma\right| > 2\alpha(G)$.

**Theorem 2.1.** *If $\Gamma$ is a König–Egerváry collection of a graph G, then*

(i) *there is a perfect matching in $G\left[\bigcup \Gamma - \bigcap \Gamma\right]$;*
(ii) $\left|\bigcup \Gamma\right| - \left|\bigcap \Gamma\right| = 2\mu\left(G\left[\bigcup \Gamma\right]\right)$;
(iii) $\alpha\left(G\left[\bigcup \Gamma\right]\right) = \alpha(G)$;
(iv) *$G\left[\bigcup \Gamma\right]$ is a König–Egerváry graph.*