ARTICLE IN PRESS

Discrete Applied Mathematics ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

The Degree/Diameter Problem for mixed abelian Cayley graphs

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ARTICLE INFO

Article history:
Received 21 November 2015
Received in revised form 21 February 2017
Accepted 19 April 2017
Available online xxxx

Keywords:
Network design
Degree/Diameter Problem
Abelian Cayley graphs

ABSTRACT

This paper investigates the upper bounds for the number of vertices in mixed abelian Cayley graphs with given degree and diameter. Additionally, in the case when the undirected degree is equal to one, we give a construction that provides a lower bound.

1. Introduction

The Degree/Diameter Problem asks for constructing the largest possible graph (in terms of the number of vertices), for a given maximum degree and a given diameter. The Degree/Diameter Problem can be formulated for directed, undirected, or mixed graphs. In the case of directed graphs, if we denote by $\overrightarrow{N}_{d,k}$ the order of the largest possible digraph that can be constructed, we get the following upper bound:

$$\overrightarrow{N}_{d,k} \leq \overrightarrow{M}_{d,k} = 1 + d + d^2 + \cdots + d^k = \begin{cases} \dfrac{d^{k+1}-1}{d-1} & \text{if } d > 1 \\ k+1 & \text{if } d = 1 \end{cases} \tag{1}$$

where d is the maximum out-degree, and k is the diameter. The number $\overrightarrow{M}_{d,k}$ is called the *(directed) Moore bound*. In the case of undirected graphs, the Moore bound becomes

$$M_{\Delta,k} = 1 + \Delta + \Delta(\Delta-1) + \cdots + \Delta(\Delta-1)^{k-1} = \begin{cases} 1 + \Delta\dfrac{(\Delta-1)^k - 1}{\Delta - 2} & \text{if } \Delta > 2 \\ 2k+1 & \text{if } \Delta = 2 \end{cases} \tag{2}$$

where Δ is the maximum degree, and k represents the diameter. These latter bounds are easily derived by just counting the number of vertices at a particular distance of any given vertex v in a [di]graph with given maximum [out-]degree and diameter [12].

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http://dx.doi.org/10.1016/j.dam.2017.04.018

Please cite this article in press as: N. López, et al., The Degree/Diameter Problem for mixed abelian Cayley graphs, Discrete Applied Mathematics (2017), http://dx.doi.org/10.1016/j.dam.2017.04.018.

In mixed graphs we have both arcs and edges. Thus we have three parameters: a maximum undirected degree r, a maximum directed out-degree z, and diameter k. The upper bound was first stated as

$$1+(r+z)+\cdots+\left[z(r+z)^{k-1}+r(r+z-1)^{k-1}\right] \tag{3}$$

but it was recently adjusted [2] to

$$M_{z,r,k}=A\frac{u_1^{k+1}-1}{u_1-1}+B\frac{u_2^{k+1}-1}{u_2-1} \tag{4}$$

where

$$\begin{aligned} v &= (z+r)^2+2(z-r)+1\\ u_1 &= \frac{z+r-1-\sqrt{v}}{2}\\ u_2 &= \frac{z+r-1+\sqrt{v}}{2}\\ A &= \frac{\sqrt{v}-(z+r+1)}{2\sqrt{v}}\\ B &= \frac{\sqrt{v}+(z+r+1)}{2\sqrt{v}}. \end{aligned}$$

Besides these general bounds given above, researchers are also interested in some particular versions of the problem, namely when the graphs are restricted to a certain class, such as the class of bipartite graphs [6], planar graphs [5,19], vertex-transitive graphs [10,16], Cayley graphs [10,16,20], Cayley graphs of abelian groups [4], or circulant graphs [7,8,13,21]. In this paper we are concerned with mixed abelian Cayley graphs.

For most of these graph classes there exist Moore-like upper bounds, which in general are smaller than the Moore bound for general graphs, although some of them are quite close to the Moore bound. For example, the best known general upper bound for undirected vertex-transitive and Cayley graphs is $M_{\Delta,k}-2$ (see [12]). On the other hand, the upper bounds for abelian Cayley graphs and digraphs are significantly smaller. Let $\overrightarrow{N}^{AC}_{d,k}$ be the number of vertices of the largest abelian Cayley digraph with degree d and diameter k; then

$$\overrightarrow{N}^{AC}_{d,k}\le\binom{k+d}{d}=\binom{k+d}{k} \tag{5}$$

(see [4]).

The generating function of the sequence $\binom{k+d}{k}$ is

$$\overrightarrow{A}_d(s)=\frac{1}{(1-s)^{d+1}}. \tag{6}$$

Alternatively, the upper bound for undirected abelian Cayley graphs is a bit more involved. It was proved in [1] that, if $\Delta=2t$, then

$$N^{AC}_{\Delta,k}\le F_{t,k}=\sum_{i=0}^{t}2^i\binom{t}{i}\binom{k}{i}. \tag{7}$$

The numbers $F_{t,k}$ of Eq. (7) are known as *Delannoy numbers* (sequence A008288 of [14]), and they arise in a variety of combinatorial and geometric problems [18]. For example, they correspond to the volume of the ball of radius $k/2$ in the L^1 metric in t dimensions [4,11]. They satisfy the recurrence

$$\begin{aligned} F_{t,k} &= F_{t-1,k}+F_{t,k-1}+F_{t-1,k-1},\ \text{with}\\ F_{t,1} &= 2t+1,\quad \text{for } t\ge 0. \end{aligned} \tag{8}$$

Other exact and asymptotic formulas are given in [7,11,17], such as

$$F_{t,k}=\sum_{i=0}^{t}\binom{t}{i}\binom{k+i}{t}=\sum_{i=0}^{t}\binom{k+i}{i}\binom{k}{t-i}. \tag{9}$$

Delannoy numbers are symmetric, hence we can swap t and k in Eq. (9), to get

$$F_{t,k}=\sum_{i=0}^{k}\binom{k}{i}\binom{t+i}{k}=\sum_{i=0}^{k}\binom{t+i}{i}\binom{t}{k-i}. \tag{10}$$