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On bounding the difference between the maximum degree and the chromatic number by a constant

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ABSTRACT

We provide a finite forbidden induced subgraph characterization for the graph class Υ_k, for all $k \in \mathbb{N}_0$, which is defined as follows. A graph is in Υ_k if for any induced subgraph, $\Delta \leq \chi - 1 + k$ holds, where Δ is the maximum degree and χ is the chromatic number of the subgraph.

We compare these results with those given in Schaudt and Weil (2015), where we studied the graph class Ω_k, for $k \in \mathbb{N}_0$, whose graphs are such that for any induced subgraph, $\Delta \leq \omega - 1 + k$ holds, where ω denotes the clique number of a graph. In particular, we give a characterization in terms of Ω_k and Υ_k of those graphs where the neighborhood of every vertex is perfect.

1. Introduction

A graph class $\mathcal{G}$ is called *hereditary* if for every graph $G \in \mathcal{G}$, every induced subgraph of G is also a member of $\mathcal{G}$. If we can describe a graph class $\mathcal{G}$ by excluding a (not necessarily finite) set of graphs as induced subgraphs, then this graph class is hereditary. A *clique* in a graph is a set of vertices of the graph that are pairwise adjacent. A maximal clique that is of largest size in a graph G is called a *maximum clique* of the graph. By $\omega(G)$ we denote the size of a maximum clique in a graph G. A *coloring* of a graph is an assignment of colors to every vertex of the graph such that adjacent vertices do not receive the same color. A coloring that uses a minimum number of colors is called an *optimal coloring*. The number of colors used in an optimal coloring of a graph G is denoted by $\chi(G)$, the so called *chromatic number*. With $V(G)$, we denote the vertex set of a graph G. The *neighborhood* of a vertex $v \in V(G)$ is denoted by $N(v)$ and comprises all vertices adjacent to v. The *degree* of v corresponds to $|N(v)|$. Finally, Δ denotes the *maximum degree* of a graph, that is, the maximum over all vertex degrees in a graph.

By Brook's Theorem [2], $\chi \leq \Delta + 1$ holds. However, it is not possible to give a *lower* bound on χ in terms of Δ only. By $K_{n,m}$, $n, m \in \mathbb{N}$, we denote the complete bipartite graph where one partition consists of n and the other of m vertices (for example, the $K_{1,4}$ can be found in Fig. 3). The set of graphs $K_{1,p}$, for all $p \in \mathbb{N}$, yields an example for an infinite family of graphs where the difference between χ and Δ is $p - 2$, and hence unbounded in this set.

Let $p \in \mathbb{N}$. With J_p, we denote the following graph. Consider a clique of size $p - 1$ and a $K_{1,p}$. Let a be a vertex in the p-partition of $K_{1,p}$ and let b be a vertex in the clique. Add the edge $\{a, b\}$ (cf. Fig. 1).

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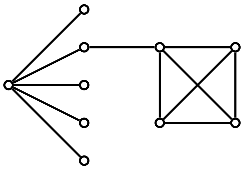

Fig. 1. The graph J_4.

In the resulting graph, Δ and χ equal $p+1$, hence the difference equals 0, but the induced $K_{1,p}$ yields a graph where the difference is $p-2$. In other words, the value of the difference between the maximum degree and the chromatic number in the host graph is not necessarily an upper bound for the respective difference in an induced subgraph. This gives rise to the following question. Which graphs guarantee that for every induced subgraph, the difference between the maximum degree and the chromatic number is at most some given number k?

In Section 2, we answer the above question in the following way. For a fixed $k \in \mathbb{N}_0$, let Υ_k be the class of graphs G for which $\Delta(H) + 1 \leq \chi(H) + k$ holds for all induced subgraphs H of G. We provide a minimal forbidden induced subgraph characterization for Υ_k. Moreover, we prove that the order of the respective minimal forbidden induced subgraph set is finite. Hence the problem of recognition of such graphs can be solved in polynomial time.

In Section 3, we connect the graph classes considered in Section 2 to the graph classes in [7]. In the latter, we considered Ω_k, where $G \in \Omega_k$ if for all induced subgraphs H of G, $\Delta(H) \leq \omega(H) + k$ holds.

In Section 4, we close this paper by giving a short summary of the results and pointing out some future directions, including generalizations of the results presented here.

Before we move on to Section 2, we give some further preliminary information. We distinguish between induced subgraphs and (partial) subgraphs. Since we deal with graph invariants, we are allowed to treat isomorphic graphs as identical. For example, if a graph G is an induced subgraph of a graph H and G is isomorphic to a graph L, then we say that L is an induced subgraph of H. A vertex is *dominating* in a graph if it is adjacent to all other vertices of the graph. In a coloring of a graph, a *color class* is the set of all vertices to which the same color is assigned to.

We denote the set of minimal forbidden induced subgraphs of Υ_k by $F(\chi, k)$, In other words, $F \in F(\chi, k)$ if and only if $F \notin \Upsilon_k$ and all proper induced subgraphs of F are contained in Υ_k. Observe that $G \in \Upsilon_k$ if and only if G is $F(\chi, k)$-free.

2. The hereditary graph class Υ_k, $k \in \mathbb{N}_0$

Let $k \in \mathbb{N}_0$. In [7], the authors introduced a family of hereditary graph classes that is quite similar to Υ_k, namely Ω_k. By definition, $G \in \Omega_k$ if every induced subgraph H of G, including G itself, obeys $\Delta(H) \leq \omega(H) + k - 1$. Let $F(\omega, k)$ denote the set of minimal forbidden induced subgraphs of Ω_k. We reword the characterization of $F(\omega, k)$ from [7] in Theorem 1.

Theorem 1 *([7]). Let G be a graph. $G \in F(\omega, k)$ if and only if the following conditions hold:*

1. *G has a unique dominating vertex v,*
2. *the intersection of all maximum cliques of G contains solely v,*
3. *$\Delta(G) = \omega(G) + k$.*

In particular, $\Delta(G) = |V(G)| - 1$ and $\omega(G) = |V(G)| - k - 1$.

Our results are primarily based on Theorem 2, whose analogism to Theorem 1 is obvious.

Theorem 2. *Let G be a graph. $G \in F(\chi, k)$ if and only if the following conditions hold:*

1. *G has a unique dominating vertex v,*
2. *each color class in every optimal coloring of $G - v$ consists of at least two vertices,*
3. *$\Delta(G) = \chi(G) + k$.*

Proof. Let $G \in F(\chi, k)$. Since G is a minimal forbidden induced subgraph, all induced subgraphs of G are contained in Υ_k except for G itself. Thus $\Delta(G) > \chi(G) + k - 1$. Choose a vertex v of maximum degree in G and let H be the graph induced in G by the vertex set $\{v\} \cup N(v)$. Observe that $H \subseteq G$ is not in Υ_k, since $\Delta(H) = \Delta(G) > \chi(G) + k - 1 \geq \chi(H) + k - 1$. Hence, $H \cong G$ by minimality of G, and thus, G contains a dominating vertex, namely v. Assume there exists a vertex $x \in V(G) \setminus \{v\}$ such that $\chi(G - x) = \chi(G) - 1$. This equality in particular holds if x is a dominating vertex. Then

$$\Delta(G - x) = \Delta(G) - 1 \geq \chi(G) + k - 1 = \chi(G - x) + k. \quad (1)$$

Thus $G - x \notin \Upsilon_k$, contradicting the minimality of G. Hence, Conditions 1 and 2 follow. Let $x \in N(v)$. Due to Condition 1, the degree of x is at most $\Delta(G) - 2$. Hence, $\Delta(G - x) = \Delta(G) - 1$. Due to Condition 2, $\chi(G - x) = \chi(G)$. Assume $\Delta(G) \geq \chi(G) + k + 1$. Then $\Delta(G - x) = \Delta(G) - 1 \geq (\chi(G) + k + 1) - 1 = \chi(G - x) + k$. That is, $G - x \notin \Upsilon_k$, a contradiction. Hence $\Delta(G) = \chi(G) + k$, and the third condition follows.

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