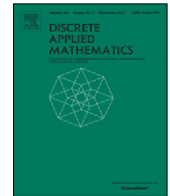




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Optimum basis of finite convex geometry

K. Adaricheva

Department of Mathematics, Hofstra University, Hempstead, New York, NY 11549, USA

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ABSTRACT

Convex geometries form a subclass of closure systems with unique criticals, or *UC*-systems. We show that the *F*-basis introduced in Adaricheva and Nation (2014) for *UC*-systems, becomes optimum in convex geometries, in two essential parts of the basis: right sides (conclusions) of binary implications and left sides (premises) of non-binary ones. The right sides of non-binary implications can also be optimized, when the convex geometry either satisfies the Carousel property, or does not have *D*-cycles. The latter generalizes a result of P.L. Hammer and A. Kogan for acyclic Horn Boolean functions. Convex geometries of order convex subsets in a poset also have tractable optimum basis. The problem of tractability of optimum basis in convex geometries in general remains to be open.

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1. Introduction

A convex geometry is a closure system with the anti-exchange axiom.

In this paper we look at representation of *finite* convex geometries by the implicational bases. This continues a series of papers [6] and [5] that translate the approaches of compact presentation of finite lattices into the realm of Horn propositional logic.

If $\Sigma = \{X_i \rightarrow Y_i : i \leq k\}$ is a set of implications defining a convex geometry, then the size of Σ is defined as $s(\Sigma) = |X_1| + \dots + |X_k| + |Y_1| + \dots + |Y_k|$. The set of implications Σ is called *optimum*, when $s(\Sigma)$ is minimum among all possible sets of implications defining convex geometry.

In this paper we address the following question: *if a convex geometry is given by a set of implications Σ , is it possible to find its optimum basis Σ_O in time polynomially dependant on $s(\Sigma)$?*

D. Maier [21] showed that the problem of finding the optimum basis for the *general closure system*, defined by a set of implications, is NP-complete, thus, the question above most likely is answered in negative. On the other hand, some special classes of closure systems may have tractable optimum bases. These are, for example, closure systems with the modular closure lattices, as shown by M. Wild [28], or quasi-acyclic closure systems, as shown by P.L. Hammer and A. Kogan [17]. Note that the latter paper deals with Horn boolean functions and their optimal CNF-representation, and there are several variations of optimization parameters. This is further discussed in K. Adaricheva and J.B. Nation [5].

E-mail address: kira.adaricheva@hofstra.edu.

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Among several types of bases introduced in [5], the F -basis becomes optimal for two parameters, when the closure system is a convex geometry: for the size of premises in non-binary part of the basis, and for the size of conclusions in a binary one. This is given in [Theorem 7](#) and [Lemma 9](#) of Section 3. In this paper, we also demonstrate three important sub-classes of convex geometries, where the conclusions of non-binary implications can also be effectively optimized, thus, where the tractable optimum basis exists: the class of geometries satisfying the n -Carousel property, order convex subsets of posets and convex geometries without D -cycles. If the first class includes all *affine* convex geometries, the third one is the generalization of *acyclic* closure systems of [17], G -geometries of M. Wild [27] and (dual) supersolvable anti-matroids of D. Armstrong [8]. We show that a convex geometry without D -cycles has the tractable optimum basis, which is exactly basis Σ_{FOE} defined in [5]. The three subclasses are addressed in Sections 4–6.

We note that all three classes differ from another tractable class, *component-quadratic* closure systems, that generalize quasi-acyclic closure systems, see E. Boros et al. [10]. We discuss this and some open problems in Section 7.

2. Preliminaries

A closure system $\mathcal{G} = \langle G, \phi \rangle$, i.e. a set G with a closure operator $\phi : 2^G \rightarrow 2^G$, is called a *convex geometry* (see [4]), if it is a zero-closed space (i.e. $\phi(\emptyset) = \emptyset$) and it satisfies the *anti-exchange axiom*, i.e.

$$x \in \phi(X \cup \{y\}) \text{ and } x \notin X \text{ imply that } y \notin \phi(X \cup \{x\}) \\ \text{for all } x \neq y \text{ in } G \text{ and all closed } X \subseteq G.$$

In this paper we consider only *finite* convex geometries, i.e. geometries with $|G| < \omega$.

It is worth noting that convex geometries are always *standard* closure systems, i.e. they satisfy property

$$\phi(\{i\}) \setminus \{i\} \text{ is closed, for every } i \in G.$$

This condition, in particular, implies $i = j$, whenever $\phi(\{i\}) = \phi(\{j\})$, for any $i, j \in G$.

Very often, a convex geometry is given by its collection of closed sets. There is a convenient description of those collections of subsets of a given finite set G , which are, in fact, the closed sets of a convex geometry on G : if $\mathcal{F} \subseteq 2^G$ satisfies

- (1) $\emptyset \in \mathcal{F}$;
- (2) $X \cap Y \in \mathcal{F}$, as soon as $X, Y \in \mathcal{F}$;
- (3) $X \in \mathcal{F}$ and $X \neq G$ implies $X \cup \{a\} \in \mathcal{F}$, for some $a \in G \setminus X$,

then $\mathcal{F}$ represents the collection of closed sets of a convex geometry $\mathcal{G} = \langle G, \phi \rangle$.

As for any closure system, the closed sets of convex geometry form a lattice, which is usually called the *closure lattice* and denoted $\text{Cl}(G, \phi)$. The closure lattices of convex geometries have various characterizations, and are usually called *locally distributive* in the lattice literature.

A reader can be referred to [12,14] and [23] for the further details of combinatorial and lattice-theoretical aspects of finite convex geometries.

If $Y \subseteq \phi(X)$, then this relation between subsets $X, Y \subseteq G$ in a closure system can be written in the form of implication: $X \rightarrow Y$. Thus, the closure system $\langle G, \phi \rangle$ can be given by the set of implications:

$$\Sigma_\phi = \{X \rightarrow Y : X \subseteq G \text{ and } Y \subseteq \phi(X)\}.$$

The set X is called the *premise*, and Y the *conclusion* of an implication $X \rightarrow Y$. We will assume that any implication $X \rightarrow Y$ is an ordered pair of non-empty subsets $X, Y \subseteq G$, and $Y \cap X = \emptyset$.

Conversely, any set of implications Σ defines a closure system: the closed sets are exactly subsets $Z \subseteq G$ that *respect* the implications from Σ , i.e., if $X \rightarrow Y$ is in Σ , and $X \subseteq Z$, then $Y \subseteq Z$. There are numerous ways to represent the same closure system by sets of implications; those sets of implications with some minimality property are called *bases*. Thus we can speak of various sorts of bases.

As in K. Adaricheva et al. [6], we will call subset $\Sigma^b = \{(A \rightarrow B) \in \Sigma : |A| = 1\}$ of given basis Σ the *binary part* of the basis. Since every convex geometry $\langle G, \phi \rangle$ is a standard closure system, the binary relation $\geq_\phi$ on G defined as:

$$a \geq_\phi b \text{ iff } b \in \phi(\{a\})$$

is a partial order. This is exactly the partial order of join irreducible elements in $L = \text{Cl}(G, \phi)$. If $a \geq_\phi b$, for $a \neq b$, then every basis of the closure system will contain an implication $a \rightarrow B$ (where b may or may not be in B). The *non-binary* part of Σ is $\Sigma^{nb} = \Sigma \setminus \Sigma^b$.

We write $|\Sigma|$ for the number of implications in Σ . Basis Σ is called *minimum*, if $|\Sigma| \leq |\Sigma^*|$, for any other basis Σ^* of the same system.

Number $s(\Sigma) = |X_1| + \dots + |X_n| + |Y_1| + \dots + |Y_n|$ is called the *size* of the basis Σ . A basis Σ is called *optimum* if $s(\Sigma) \leq s(\Sigma^*)$, for any other basis Σ^* of the system. Similarly, one can define $s_L(\Sigma) = |X_1| + \dots + |X_n|$, the L -size, and $s_R(\Sigma) = |Y_1| + \dots + |Y_n|$, the R -size, of a basis Σ . The basis will be called *left-side optimum* (resp. *right-side optimum*), if $s_L(\Sigma) \leq s_L(\Sigma^*)$ (resp. $s_R(\Sigma) \leq s_R(\Sigma^*)$), for any other basis Σ^* .

Now we recall the major theorem of V. Duquenne and J.L. Guigues about the canonical basis [16], also see N. Caspard and B. Monjardet [11].

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