Discrete Applied Mathematics ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

On some domination colorings of graphs

Guillaume Bagan [a,b,*], Houcine Boumediene-Merouane [c], Mohammed Haddad [a,b], Hamamache Kheddouci [a,b]

[a] Lab. LIRIS, UMR CNRS 5205, University of Lyon, F-69003, France
[b] University of Claude Bernard Lyon 1, 43 Bd du 11 Novembre 1918, F-69622, Villeurbanne, France
[c] Lab. LAMDARO, University of Saad Dahlab, B.P. 270, Blida, Algeria

ARTICLE INFO

Article history:
Received 6 June 2016
Received in revised form 20 May 2017
Accepted 13 June 2017
Available online xxxx

Keywords:
Dominator coloring
Total dominator coloring
Dominated coloring
Strong coloring
Strict strong coloring

ABSTRACT

In this paper, we are interested in four proper vertex colorings of graphs, with additional domination property. In the *dominator colorings*, *strong colorings* and *strict strong colorings* of a graph G, every vertex has to dominate at least one color class. Conversely, in the *dominated colorings* of G, every color class has to be dominated by at least one vertex. We study arbitrary graphs as well as P_4-sparse graphs, P_5-free graphs, bounded treewidth graphs and claw-free graphs.

1. Introduction

Let $G = (V, E)$ be a graph such that V is the vertex set and E is the edge set. A proper vertex k-coloring of a graph $G = (V, E)$ is a mapping $\lambda : V \to \{1, 2, \dots, k\}$ such that any two adjacent vertices receive different colors. This problem is equivalent to the problem of partitioning the vertex set of G into k independent sets $\{C_1, C_2, \dots, C_k\}$ where $C_i = \{x \in V : \lambda(x) = i\}$. The smallest integer k such that G has a k-coloring is the chromatic number $\chi(G)$ of G. The set of all vertices colored with the same color is called a *color class*.

Graph coloring is one of the major, well-known and well-studied problems in graph theory. A remarkable number of strong theorems and conjectures associated with graph coloring are being published forming a rich literature [35]. In addition to classical vertex coloring, there exist some variations like edge-coloring or list-coloring. Furthermore, graph colorings and domination problems are often in relation [10]. Recently, a whole new family of graph colorings inducing domination relations between vertices and color classes have been proposed. These colorings are strong coloring, dominator coloring, strict strong coloring, also called total dominator coloring, and dominated coloring.

1.1. Paper organization

In this paper, we survey the most important results related to this family of graph colorings and present new results. After preliminaries and definitions given in following subsections, we begin our study with a well known graph, the Petersen graph in Section 2 and we study the existence of graphs with fixed parameters. In Section 3, we study P_4-sparse graphs. We are

* Corresponding author.
E-mail address: guillaume.bagan@liris.cnrs.fr (G. Bagan).

http://dx.doi.org/10.1016/j.dam.2017.06.013

interested in Section 4 in P_5-free graphs. In Section 5, we give linear time algorithms for graphs of bounded treewidth. We study claw-free graphs in Section 6 respectively. We conclude the paper with a summary of complexity results in various graph classes and some open questions.

1.2. Basic notions of graphs

Graphs considered in this paper are simple and undirected. Colorings are proper vertex colorings. $[k, l]$ denotes the set $\{k, \ldots, l\}$. Given a graph $G = (V, E)$, the *complement graph* $\bar{G} = (V, \bar{E})$ is the graph on the same set of vertices such that two vertices of $\bar{G}$ are adjacent if and only if they are nonadjacent in G. The *line graph* of G is the graph where the vertices are the edges of G and two vertices are adjacent if and only if their corresponding edges are adjacent in G. Given a set $X \subseteq V$, we denote by $G[X]$ the subgraph of G induced by X. A *component* of a graph G is an induced subgraph of G that is connected and maximal with this property. The *open* and *closed neighborhoods* of a vertex $x \in V$ are denoted by $N(x)$ and $N[x]$, respectively. The open and closed neighborhoods of a set $X \subseteq V$ are respectively $N(X) = \cup_{x \in X} N(x)$ and $N[X] = N(X) \cup X$. The degree of a vertex $x \in V$ is the cardinality of its open neighborhood and the *maximum degree* of a graph G is denoted by $\Delta(G)$. A clique is any set $K \subseteq V$ such that each two vertices of K are adjacent. An independent set is any set $S \subseteq V$ such that no edge of G has its two end vertices in S. The *clique number* (resp. *independent set number*) of G, denoted $\omega(G)$ (resp. $\alpha(G)$), is the cardinality of a maximum clique (resp. in dependent set) in G. We denote by P_n the path on n vertices and by C_n the cycle on n vertices. The complete graph on n vertices is denoted by K_n and the complete graph of order 3 is called a *triangle*. The complete bipartite graph with classes of orders r and s is denoted by $K_{r,s}$. A *star* is a graph $K_{1,k}$ with $k \geq 1$. A *bi-star* is a graph formed by two stars by adding an edge between the center vertices. Given any graph H, a graph G is *H-free* if it does not have any induced subgraph isomorphic to H. A *tree* is a connected acyclic graph. A graph is *chordal* if it has no induced cycles of length more than three. A vertex of degree one is a *leaf* and its neighbor is a *stem*.

1.3. Colorings and dominating sets

A set $S \subseteq V$ of G is a dominating set of G if every vertex of $V \setminus S$ has a neighbor in S, *i.e.* $N[S] = V$. A variant of domination is the *total domination* where $N(S) = V$. The domination (resp. total domination) number $\gamma(G)$ (resp. $\gamma_t(G)$) is the minimum cardinality of a dominating (resp. total dominating) set of G. Obviously, $\gamma(G) \leq \gamma_t(G)$. A $\gamma(G)$-set (resp. $\gamma_t(G)$-set) is a dominating (resp. total dominating) set of G with minimum cardinality.

We say that a vertex $u \in V$ *dominates* a color class c if u is the unique vertex of c or it is adjacent to all vertices of c. We say that u totally dominates a color class c if u dominates c and u is not contained in c. A *dominator coloring* of a graph G is a proper coloring of G such that every vertex of G dominates at least one color class. Empty color classes are not allowed. A *strong coloring* of a graph G is a proper coloring of G such that every vertex of G totally dominates at least one color class. Empty color classes are allowed. A *strict strong coloring* or *total dominator coloring* of a graph G with no isolated vertices is a strong coloring of G where no empty color class is allowed. A *dominated coloring* is a proper coloring where every color class is totally dominated by at least one vertex. The minimum number of colors required for dominator, strong, strict strong and dominated colorings of G are called *dominator*, *strong*, *strict strong* and *dominated chromatic numbers*, denoted by $\chi_d(G)$, $\chi_s(G)$, $\chi_{ss}(G)$ and $\chi_{dom}(G)$, respectively.

A $\chi(G)$-coloring (resp. $\chi_d(G)$-coloring, $\chi_{dom}(G)$-coloring, $\chi_s(G)$-coloring, $\chi_{ss}(G)$-coloring) is a proper coloring (resp. dominator, dominated, strong, strict strong) coloring of G using $\chi(G)$ (resp. $\chi_d(G)$, $\chi_{dom}(G)$, $\chi_s(G)$, $\chi_{ss}(G)$) colors.

Consider a k-coloring of the graph G and let $V_1, V_2, \ldots, V_k$ be the color classes. A color class of cardinality k is said *k-class*. A vertex $v \in V_i$ is called *solitary* if V_i is a 1-class. We denote by C_S the set of 1-classes, by C_D the set of dominated k-classes and by C_N the set of non-dominated k-classes, with $k \geq 2$. The set of vertices of elements of C_S, C_D and C_N are denoted by V_S, V_D and V_N, respectively. Clearly, $|C_S| = |V_S|$ and $\{C_S, C_D, C_N\}$ (resp. $\{V_S, V_D, V_N\}$) is a partition of $\{V_1, V_2, \ldots, V_k\}$ (resp. of V).

The problem of dominator coloring was introduced by Hedetniemi et al. in 2006 [22] and studied further in [3,2,20,18,19,9,34].

It is easy to check Inequalities (1) due to Gera et al. [20]. Indeed, consider a graph $G = (V, E)$ and a $\chi_d(G)$-coloring. The set of vertices obtained by taking exactly one vertex from each dominated color class is a dominating set. Now, consider $S \subseteq V$ a $\gamma(G)$-set. We obtain a dominator coloring of G by giving a unique color to each vertex of S and at most $\chi(G)$ colors to vertices of $V \setminus S$.

$$\max\{\gamma(G), \chi(G)\} \leq \chi_d(G) \leq \gamma(G) + \chi(G). \tag{1}$$

In [39], Zverovich introduced the strong coloring of graphs wherein an empty color class is allowed. If the empty class exists then it is dominated by all vertices. Inequalities (2) follow.

$$\chi(G) \leq \chi_s(G) \leq \chi(G) + 1. \tag{2}$$

Please cite this article in press as: G. Bagan, et al., On some domination colorings of graphs, Discrete Applied Mathematics (2017), http://dx.doi.org/10.1016/j.dam.2017.06.013.