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Almost all regular graphs are normal

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ABSTRACT

In 1999, De Simone and Körner conjectured that every graph without induced $C_5, C_7, \overline{C}_7$ contains a clique cover $\mathcal{C}$ and a stable set cover $\mathcal{I}$ such that every clique in $\mathcal{C}$ and every stable set in $\mathcal{I}$ have a vertex in common. This conjecture has roots in information theory and became known as the Normal Graph Conjecture. Here we prove that all C_4-free graphs of bounded maximum degree and sufficiently large odd girth (linear in the maximum degree) are normal. This is used to prove that for every fixed d, random d-regular graphs are a.a.s. normal.

1. Introduction

A graph G is said to be *normal* if it contains a set $\mathcal{C}$ of cliques and a set $\mathcal{I}$ of stable sets with the following properties:

(1) $\mathcal{C}$ is a cover of G, i.e., every vertex in G belongs to one of the cliques in $\mathcal{C}$;
(2) $\mathcal{I}$ is a cover of G, i.e., every vertex in G belongs to one of the stable sets in $\mathcal{I}$;
(3) Every clique in $\mathcal{C}$ and every stable set in $\mathcal{I}$ have a vertex in common.

Clearly, a graph is normal if and only if its complement is normal. This property is reminiscent on the notion of perfect graphs. Namely, normality is one of the basic properties that every perfect graph satisfies. Of course, normality is much weaker condition since every odd cycle of length at least 9 is normal.

The importance of normality of graphs lies in its close relationship to the notion of graph entropy, one of central concepts in information theory; see Csiszár and Körner [2] or [3,7,8].

A set $\mathcal{C}$ of edges of a graph G is a *star cover* of G if every vertex of positive degree in G is incident with an edge in $\mathcal{C}$ and each component formed from the edges in $\mathcal{C}$ is a star (a graph isomorphic to $K_{1,t}$ for some $t \geq 1$). In the definition of normality, one may ask that the clique cover $\mathcal{C}$ is minimal. Note that a minimal clique cover in a triangle-free graph is the same as a star cover.

A star cover $\mathcal{C}$ of a graph G is *nice* if every odd cycle in G contains at least 3 vertices whose incident edges in the cycle are either both or none in $\mathcal{C}$. For triangle-free graphs, De Simone and Körner [4] proved the following relationship between normality and existence of nice star covers.

Theorem 1.1. *A triangle-free graph is normal if and only if it has a nice star cover.*

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Let Q be a cycle of a graph G and $\mathcal{C}$ be a star cover of G. Then a vertex v of Q is a *good vertex* (with respect to $\mathcal{C}$) if the two edges of Q incident with v are either both in $\mathcal{C}$ or none is in $\mathcal{C}$. We define when a degree-2 vertex on a path is a *good vertex* in the same way. Let h be the number of components of $Q \cap \mathcal{C}$. Note that the number of good vertices in the cycle Q is equal to $|Q| - 2h$. Hence, we have the following observation.

Observation 1.2. *Let Q be an odd cycle. Then the number of good vertices of Q is odd.*

This observation shows that a star cover $\mathcal{C}$ is nice if and only if every odd cycle Q of G has at least two good vertices.

In their inspiring work, De Simone and Körner [4] proposed the following conjecture:

Conjecture 1.3. *A graph is normal if it contains no induced cycles of length 5 or 7 and does not contain the complement of C_7 as an induced subgraph.*

This conjecture has been verified for a very small class of graphs, including line graphs of cubic graphs [9], minimal asteroidal triple graphs [11], and a specific class of circulants [12]. See also [10] and [13]. We proved this conjecture for triangle-free subcubic graphs in [1].

Counterexamples to Conjecture 1.3 have been announced in [5]. They are constructed from certain random graphs. However, one of the main results of this paper (Theorem 3.3) shows that random d-regular graphs are asymptotically almost surely (a.a.s.) normal.

To prove Theorem 3.3, we first show that all C_4-free graphs with maximum degree at most k and odd girth at least $16k-19$ are normal. See Theorem 2.1.

Remark 1.4. The proof technique of [1] would enable us to prove Theorem 2.1 without excluding 4-cycles. However, since our main motivation is Theorem 3.3, it does not seem to be worth the additional efforts.

2. Graphs with maximum degree k

Here we extend the method that was used in [1] for cubic graphs to graphs with vertices of degrees more than 3. The proof is by induction on $\Delta(G)$ and gives a bound on the odd girth that is linear in $\Delta(G)$. We will refrain from trying to optimize this bound since it is too far from the conjectured value in Conjecture 1.3.

Let us define a star cover $\mathcal{C}$ of a graph G by using the following procedure. We assume that G has no isolated vertices and is C_4-free.

(1) Let $k = \Delta(G)$, $G_k = G$, $\mathcal{C} = \emptyset$ and let $s = k$.
(2) Let F_s be a maximum set of vertex-disjoint s-stars in G_s such that $G'_s = G_s \setminus V(F_s)$ has no isolated vertices apart from those that are already present in G_s. (When $s = k$, there are none, but they may arise later when this step is repeated with $s < k$.) Add F_s to $\mathcal{C}$.
(3) Let U'_s be the set of vertices u' whose degree in G'_s is equal to 1 and whose neighbor is adjacent to a vertex w of degree s in G'_s (such a vertex u' prevented us from adding the star centered at w to F_s since u' would then become an isolated vertex). Let U''_s be the neighbors in G'_s of vertices in U'_s and let $U_s = U'_s \cup U''_s$. Add all $U'_s - U''_s$ edges to $\mathcal{C}$ and let $G_{s-1} = G'_s \setminus U_s$. Note that since G is C_4-free, $\Delta(G_{s-1}) \leq s-1$ and that the last step may give rise to some isolated vertices in G_{s-1}. Also note that such an isolated vertex is necessarily adjacent to two or more vertices in U''_s and possibly to some in U''_t for $t < s$.
(4) If $s > 3$ then decrease s by one and go to (2).
(5) Note that the subgraph G_2 consists of paths and cycles, possibly including some isolated vertices. Let V_2 be the set of isolated vertices in G_2. For each vertex in V_2, choose one of its neighbors in U''_s where s is as small as possible. Add the edge joining them into $\mathcal{C}$. If we have a pattern of edges in $\mathcal{C}$ like the one shown at the top of Fig. 1, we change it to the cover shown at the bottom of Fig. 1.
(6) Note that since G is C_4-free the subgraph $G'' = G_2 \setminus V_2$ consists of non-trivial paths and cycles. Cover the vertices of each path or cycle in G'' with 2-stars and at most 2 single edges.

Using the above construction of a star cover $\mathcal{C}$, we can prove the following.

Theorem 2.1. *Every C_4-free graph G with maximum degree $k \geq 3$ and odd girth at least $16k-19$ is normal.*

Let us consider construction (1)–(6) of the star cover $\mathcal{C}$ described above. Furthermore, note that if Q is a cycle in G and $v \in V(Q) \cap F_k$ where $k = \Delta(G)$, then either v is a good vertex of Q, or there is an edge $vu \in E(Q) \cap E(F_k)$ such that u is a good vertex in Q.

For $\mathcal{C} \subseteq E(G)$, we define a *$\mathcal{C}$-alternating path* (*cycle*) as a path (cycle) whose edges alternate between edge in $\mathcal{C}$ and $E(G) \setminus \mathcal{C}$. We denote by $S(v)$ the set of edges incident with v.

Lemma 2.2. *Let $P = v_0v_1v_2v_3 \ldots v_{2r}v_{2r+1}$ be a $\mathcal{C}$-alternating path such that $v_0v_1 \in \mathcal{C}$. Let $T_1(s) = \{v_{2i}v_{2i+1} : v_{2i}v_{2i+1} \in S(v_{2i+1}) \subseteq F_s,\ 0 \leq i \leq r\}$, $T_2(s) = \{v_{2i}v_{2i+1} : v_{2i} \in U''_s, v_{2i+1} \in U'_s,\ 0 \leq i \leq r\}$ and $T_3(s) = \{v_{2i}v_{2i+1} : v_{2i} \in U''_s, v_{2i+1} \in V_2,\ 0 \leq i \leq r\}$. If $v_0v_1 \in T_1(s) \cup T_2(s) \cup T_3(s)$, then:*