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The numbers of repeated palindromes in the Fibonacci and Tribonacci words

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ABSTRACT

The Fibonacci word $\mathbb{F}$ is the fixed point beginning with a of morphism $\sigma(a) = ab$ and $\sigma(b) = a$. Since $\mathbb{F}$ is uniformly recurrent, each factor ω appears infinitely many times in the sequence which is arranged as ω_p (the pth occurrence of ω, $p \geq 1$). Here we distinguish $\omega_p \neq \omega_q$ if $p \neq q$. In this paper, we give an algorithm for counting the number of repeated palindromes in $\mathbb{F}[1, n]$ (the prefix of $\mathbb{F}$ of length n). That is the number of the pairs (ω, p), where ω is a palindrome and $\omega_p \prec \mathbb{F}[1, n]$. We also get explicit expressions for some special n such as $n = f_m$ (the mth Fibonacci number). Similar results are also given for the Tribonacci word, the fixed point beginning with a of morphism $\tau(a) = ab$, $\tau(b) = ac$ and $\tau(c) = a$.

1. Introduction

The Fibonacci word $\mathbb{F}$ is the fixed point beginning with a of morphism $\sigma(a) = ab$ and $\sigma(b) = a$. We define $F_m = \sigma^m(a)$ for $m \geq 0$, $F_{-2} = \varepsilon$ (empty word), $F_{-1} = b$. Then $F_0 = a$, $F_m = F_{m-1}F_{m-2}$ for $m \geq 1$. We call $|F_m| = f_m$ the mth Fibonacci number for $m \geq -2$. Here $|\omega|$ means the length of ω. Similarly, the Tribonacci word $\mathbb{T}$ is the fixed point beginning with a of morphism $\tau(a) = ab$, $\tau(b) = ac$ and $\tau(c) = a$. We define $T_m = \tau^m(a)$ for $m \geq 0$, $T_{-2} = \varepsilon$, $T_{-1} = c$. Then $T_0 = a$, $T_1 = ab$, $T_m = T_{m-1}T_{m-2}T_{m-3}$ for $m \geq 2$. We call $|T_m| = t_m$ the mth Tribonacci number for $m \geq -2$.

Let $\rho = x_1 \cdots x_n$ be a finite word (or $\rho = x_1x_2 \cdots$ be an infinite word). For any $1 \leq i \leq j \leq n$, we define $\rho[i, j] = x_ix_{i+1} \cdots x_{j-1}x_j$, $\rho[i] = \rho[i, i] = x_i$ and $\rho[i, i-1] = \varepsilon$.

We say that ν is a prefix (resp. suffix) of a word ω if there exists word u such that $\omega = \nu u$ (resp. $\omega = u\nu$), $|u| \geq 0$, which denoted by $\nu \lhd \omega$ (resp. $\nu \rhd \omega$). In this case, we write $\nu^{-1}\omega = u$ (resp. $\omega\nu^{-1} = u$), where ν^{-1} is the inverse word of ν such that $\nu\nu^{-1} = \nu^{-1}\nu = \varepsilon$. Obviously, $\mathbb{F}[1, n]$ (resp. $\mathbb{T}[1, n]$) is the prefix of $\mathbb{F}$ (resp. $\mathbb{T}$) of length n.

Let ω be a factor of $\mathbb{F}$, denoted by $\omega \prec \mathbb{F}$. The word $\mathbb{F}$ is uniformly recurrent, i.e., each factor ω occurs infinitely often and with bounded gaps between consecutive occurrences [1]. We arrange them in the sequence $\{\omega_p\}_{p\geq 1}$, where ω_p denotes the pth occurrence of ω. We denote by $P(\omega, p)$ the position of the last letter (also called the end position) of ω_p. We denote the gap between ω_p and ω_{p+1} by $G_p(\omega)$. Concretely, for $p \geq 1$, let $\omega_p = x_{i+1} \cdots x_{i+n}$ and $\omega_{p+1} = x_{j+1} \cdots x_{j+n}$. Then when $i+n < j$, $G_p(\omega) = x_{i+n+1} \cdots x_j$; when $i+n = j$, $G_p(\omega) = \varepsilon$; when $i+n > j$, $G_p(\omega) = (x_{j+1} \cdots x_{i+n})^{-1}$. The sequence $\{G_p(\omega)\}_{p\geq 1}$ is called the gap sequence of factor ω.

A palindrome is a finite word that reads the same backwards as forwards. Let $\mathcal{P}_F$ (resp. $\mathcal{P}_T$) be all palindromes occurring in $\mathbb{F}$ (resp. $\mathbb{T}$). Some previous research has been done on "rich word", which is based on the number of distinct palindromes. A finite word ω is rich if and only if ω contains exactly $|\omega| + 1$ distinct palindromes (including the empty word). An infinite

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Table 1
The first few Fibonacci words and singular words.

m	-2	-1	0	1	2	3	4	5
F_m	ε	b	a	ab	aba	$abaab$	$abaababa$	$abaababaabaab$
K_m	ε	a	b	aa	bab	$aabaa$	$babaabab$	$aabaababaabaa$

word is rich if and only if all of its factors are rich. Droubay–Justin–Pirillo [4] proved that episturmian sequences are rich. Therefore, as special cases, $\mathbb{F}$ and $\mathbb{T}$ are rich. Thus the number of distinct palindromes in $\mathbb{F}[1, n]$ (resp. $\mathbb{T}[1, n]$) is $n + 1$ for all n.

In this paper, we consider the numbers of repeated palindromes in $\mathbb{F}[1, n]$ and $\mathbb{T}[1, n]$. Denote

$$A(n) = \#\{(\omega, p) \mid \omega \in \mathcal{P}_F, \omega_p \prec \mathbb{F}[1, n]\} \text{ and } B(n) = \#\{(\omega, p) \mid \omega \in \mathcal{P}_T, \omega_p \prec \mathbb{T}[1, n]\}.$$

The research on counting the repeated palindromes is not rich. From our knowledge, it seems we are the first to study this problem. In related fields, the numbers of special types of factors have been investigated in recent years, such as squares, cubes, r-powers, palindromes, runs, Lyndon factors, etc. See [3,4,6–8,11–18].

The main difficulty of this problem is twofold: (1) The positions of all occurrences for all palindromes are not easy to be determined. In this paper, we overcome this difficulty by using the "gap sequence" properties of $\mathbb{F}$ and $\mathbb{T}$, which we introduced and studied in [9,10]. (2) Taking $\mathbb{F}$ for instance, by the gap sequence property of $\mathbb{F}$, we can find out all distinct palindromes in $\mathbb{F}[1, n]$. We can also count the number of occurrences of each palindrome. So the summation of these numbers are the numbers of repeated palindromes in $\mathbb{F}[1, n]$. But this method is complicated. We overcome this difficulty by studying the relations among positions of each ω_p, and establishing the recursive structure of $\mathcal{P}_F$ in Section 3. The similar structure of $\mathcal{P}_T$ is established in Section 6.

Using the gap sequence properties and recursive structures, we give algorithms for counting $A(n)$ and $B(n)$ in Sections 4 and 7, respectively. We also get explicit expressions for some special n, such as: for $m \geq 0$,

$$\begin{cases} A(f_m) = \frac{m-3}{5} f_{m+2} + \frac{m-1}{5} f_m + m + 3, \\ B(t_m) = \frac{m}{22}\left(10t_m + 5t_{m-1} + 3t_{m-2}\right) + \frac{1}{22}\left(-23t_m + 12t_{m-1} - 5t_{m-2}\right) + m + \frac{3}{2}. \end{cases}$$

We think this method for counting the repeated palindromes is suitable for the m-bonacci word, and even suitable for sturmian sequences, episturmian sequences etc. But now we only have the gap sequence properties of $\mathbb{F}$ and $\mathbb{T}$. As a final remark, we establish the cylinder structures and chain structures of $\mathcal{P}_F$ and $\mathcal{P}_T$ in Section 8. Using them, we prove some known results.

2. Preliminaries of the Fibonacci word

Denote by δ_m the last letter of F_m for $m \geq -1$, then $\delta_m = a$ if and only if m is even. The mth kernel word of $\mathbb{F}$ is defined as $K_m = \delta_{m+1} F_m \delta_m^{-1}$ for $m \geq -2$, which is also called singular word in Wen–Wen [19] (see Table 1).

By [19], all kernel words are palindromes and $K_m = K_{m-2} K_{m-3} K_{m-2}$ for all $m \geq 2$. Let $Ker(\omega)$ be the maximal kernel word occurring in factor ω. Then by Theorem 1.9 in Huang–Wen [9], $Ker(\omega)$ occurs in ω only once.

Property 2.1 (*Theorem 2.8 in [9]*). *$Ker(\omega_p) = Ker(\omega)_p$ for all $\omega \in \mathbb{F}$ and $p \geq 1$.*

This means, let $Ker(\omega) = K_m$, then the maximal kernel word occurring in ω_p is just the pth occurrence of K_m, denoted by $K_{m,p}$. For instance, $Ker(aba) = b$, $(aba)_3 = \mathbb{F}[6, 8]$, $(b)_3 = \mathbb{F}[7]$, so $Ker((aba)_3) = (b)_3$, $(aba)_3 = a(b)_3 a$.

In [9], we used an equivalent notion of "gap" called "return word", which was introduced by Durand [5]. Let $r_p(\omega)$ be the pth return word of ω, then $r_p(\omega) = \omega G_p(\omega)$. Using this relation, we rewrite Theorem 2.11 in [9] as below.

Property 2.2. *The gap sequence $\{G_p(\omega)\}_{p \geq 1}$ is $\mathbb{F}$ over the alphabet $\{G_1(\omega), G_2(\omega)\}$ for all $\omega \in \mathbb{F}$.*

3. The recursive structure of $\mathcal{P}_F$

We establish the recursive structures of $\mathcal{P}_F$ in this section. By Proposition 3.4 in [9], we have

Property 3.1. *For $m \geq -1$ and $p \geq 1$, the end position of the pth occurrence of K_m is*

$$P(K_m, p) = p f_{m+1} + (\lfloor \phi p \rfloor + 1) f_m - 1.$$

Here $\phi = \frac{\sqrt{5}-1}{2}$ and $\lfloor \alpha \rfloor$ is the largest integer not more than α. In particular, $P(a, p) = p + \lfloor \phi p \rfloor$, $P(b, p) = 2p + \lfloor \phi p \rfloor$ and $P(aa, p) = 3p + 2\lfloor \phi p \rfloor + 1$.