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Rainbow connection number and graph operations☆

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ABSTRACT

A path in an edge-colored graph G is *rainbow* if no two edges of the path are colored the same. An edge-colored graph G is *rainbow connected* if every two distinct vertices are connected by a rainbow path. The *rainbow connection number* $rc(G)$ of G is the smallest number of colors that are needed in order to make G rainbow connected. In this paper, we study bounds of rainbow connection number of some graph operations, such as the union of two graphs, adding edges, deleting edges, and adding vertices and edges. Moreover, we also study the following extremal problem. Let k and n be two integers such that $1 \leq k \leq \ell < n$. Find the smallest integer $f(n, k, \ell)$ such that for each graph G of order n and diameter k, there exists an edge set $F \subseteq E(\overline{G})$ satisfying $|F| \leq f(n, k, \ell)$ and $rc(G + F) \leq \ell$.

1. Introduction

All graphs in this paper are undirected, finite, and simple. We refer to the book [5] for notation and terminology not described here. The *distance* between two vertices x and y in G, denoted by $d_G(x, y)$, is the number of edges in the shortest path between them. The *eccentricity* of a vertex x, denoted by $ecc_G(x)$, is $max_{y \in V(G)} d_G(x, y)$. The *radius* and *diameter* of G, denoted by $rad(G)$ and $diam(G)$, are $min_{x \in V(G)} ecc_G(x)$ and $max_{x \in V(G)} ecc_G(x)$, respectively. A vertex u is a *center* if $ecc_G(u) = rad(G)$.

A path in an edge-colored graph G, where adjacent edges may have the same color, is *rainbow* if no two edges of the path are colored the same. An edge-colored graph G is *rainbow connected* if every two distinct vertices are connected by a rainbow path. An edge-coloring under which G is rainbow connected is *rainbow-connected* of G. The *rainbow connection number* of G, denoted by $rc(G)$, is the smallest number of colors that are needed in order to make G rainbow connected. It is easy to see that $diam(G) \leq rc(G)$ for any connected graph G.

The rainbow connection number was introduced by Chartrand, Johns, McKeon, and Zhang in [8]. It has applications in transferring information of high security in multicomputer networks. We refer the reader to [6,16] for details.

Chakraborty, Fischer, Matsliah, and Yuster [6] investigated the hardness and algorithms for the rainbow connection number, and showed that given a graph G, deciding if $rc(G) = 2$ is NP-complete. For further algorithmic results, we refer the interest reader to [1,7,9].

Bounds of the rainbow connection number of a graph have also been studied in terms of other graph parameters, for example, radius and diameter, etc. [2,11–13]. Extremal problems have been studied by [4,14,17,18]. Graph operations have been studied by [3,10,15].

In [3], Basavaraju, Chandran, Rajendraprasad, and Ramaswamy studied the rainbow connection number of graph power and graph products and obtained the following results.

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Theorem 1 (*[3]*). *For a connected graph G, let G^k be the kth power of G. Then $rad(G^k) \le rc(G^k) \le 2rad(G^k) + 1$. The upper bound is tight up to an additive constant of* 1.

Theorem 2 (*[3]*). *If G and H are two connected, non-trivial graphs, then*

$$rad(G\Box H) \le rc(G\Box H) \le 2rad(G\Box H).$$

The bounds are tight.

Theorem 3 (*[3]*). *Given two non-trivial graphs G and H such that G is connected, we have the following:*
(1) *If $rad(G \circ H) \ge 2$, then $rad(G \circ H) \le rc(G \circ H) \le 2rad(G \circ H)$. The bounds are tight.*
(2) *If $rad(G \circ H) = 1$, then $1 \le rc(G \circ H) \le 3$. The bound are tight.*

In [10], Gologranca, Mekiš, and Peterin studied the rainbow connection number on direct, strong, and lexicographic product by splitting a graph G into two spanning subgraphs. Because of the need to introduce several new concepts, we do not detail their results, and we refer the interest reader to [10] for details.

In [15], Li, and Sun studied the rainbow connection number of line graphs, and showed the following results.

Observation 4 (*[15]*). *If G is a connected graph and $\{E_i\}_{i\in[t]}$ is a partition of the edge set of G into connected subgraphs $G_i = G[E_i]$, then $rc(G) \le \sum_{i=1}^{t} rc(G_i)$.*

Theorem 5 (*[15]*). *If G is a connected graph, $\mathcal{T}$ is a set of t edge-disjoint triangles that cover all but n inner vertices of G and c is the number of components of the subgraph $G[E(\mathcal{T})]$, then $rc(L(G)) \le t + n + c$. Moreover, the bound is sharp.*

In [8], Chartrand, Johns, McKeon, and Zhang determined the rainbow connection number of complete bipartite graphs, which is useful for our proof.

Theorem 6 (*[8]*). *For integers s and t with $2 \le s \le t$, $rc(K_{s,t}) = min\{\lceil \sqrt[s]{t} \rceil, 4\}$.*

In [13], Li, Li, and Liu investigated the rainbow connection number of graphs with diameter two, which is useful for our proof.

Theorem 7 (*[13]*). *If G is a bridgeless graph with diameter* 2, *then $rc(G) \le 5$.*

For integers n and ℓ, let $t(n, k)$ denote the minimum number of edges of graphs with order n and rainbow connection number at most k. The graph parameter $t(n, k)$ was introduced by Schiermeyer in [18], and investigated by Bode and Harborth in [4], Li, Li, Sun, and Zhao in [14], and Lo in [17]. The following results are useful for our proof.

Theorem 8 (*[18]*). $t(n, 2) \le (n + 1)\lfloor \log_2 n \rfloor - 2^{\lfloor \log_2 n \rfloor} - 2$.

Theorem 9 (*[17]*). *If $k, n \ge 3$, then $t(n, k) = \lceil \frac{k(n-2)}{k-1} \rceil$.*

This paper is organized as follows. In Section 2, we study the rainbow connection number of the union of graphs. In Section 3, we study how the rainbow connection number of a graph behaves under edge addition and deletion. In Section 4, we study the method of adding new vertices and edges to a graph G such that the rainbow connection number of the new graph is not more than the rainbow connection number of G.

2. The union of graphs

We use P_n to denote a path with n vertices. A path P is called a *u-v path*, denoted by P_{uv}, if u and v are the endpoints of P. Let G be a graph, and let $X \subseteq V(G)$. For any integer $k \ge 1$, the *k-step open neighborhood* $N_G^k(X)$ is $\{y \in V(G) : d_G(X, y) = k\}$, where $d_G(X, y) = \min\{d_G(x, y) : x \in X\}$. We simply write $N_G(X)$ for $N_G^1(X)$ and $N_G^k(x)$ for $N_G^k(\{x\})$. Similarly, the *k-step closed neighborhood* $N_G^k[X]$ is $\{y \in V(G) : d(X, y) \le k\}$. We simply write $N_G[X]$ for $N_G^1[X]$ and $N_G^k[x]$ for $N_G^k[\{x\}]$ if $X = \{x\}$. For any two subsets X and Y of $V(G)$, let $E_G[X, Y]$ denote $\{xy : x \in X, y \in Y, xy \in E(G)\}$. We simply write $E_G[x, Y]$ for $E_G[X, Y]$ if $X = \{x\}$.

For simplicity, we use (G, c) to denote a graph with edge-coloring c. For a subgraph H in (G, c), we use $c(H)$ to denote the set of colors used by H, that is, $c(H) = \{c(e) : e \in H\}$.

Recall that the *union* of simple graph G and H is the graph $G \cup H$ with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. If G and H are disjoint, we refer to their union as a *disjoint union*, and denote it by $G + H$. If $G_1, G_2, \dots, G_n$ are isomorphic, then we denote $G_1 + G_2 + \cdots + G_n$ by nG_1.

Theorem 10. *Let G and H be two connected graphs. If $V(G) \cap V(H) \ne \emptyset$, then*

$$rc(G \cup H) \le rc(G) + rc(H).$$

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