ARTICLE IN PRESS

Discrete Applied Mathematics ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Ramsey numbers of 4-uniform loose cycles

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ARTICLE INFO

Article history:
Received 1 December 2016
Accepted 30 April 2017
Available online xxxx

Keywords:
Ramsey number
Uniform hypergraph
Loose path
Loose cycle

ABSTRACT

Gyárfás, Sárközy and Szemerédi proved that the 2-color Ramsey number $R(\mathcal{C}_n^k, \mathcal{C}_n^k)$ of a k-uniform loose cycle $\mathcal{C}_n^k$ is asymptotically $\frac{1}{2}(2k-1)n$, generating the same result for $k = 3$ due to Haxell et al. Concerning their results, it is conjectured that for every $n \geq m \geq 3$ and $k \geq 3$,

$$R(\mathcal{C}_n^k, \mathcal{C}_m^k) = (k-1)n + \left\lfloor \frac{m-1}{2} \right\rfloor.$$

In 2014, the case $k = 3$ is proved by the authors. Recently, the authors showed that this conjecture is true for $n = m \geq 2$ and $k \geq 8$. Their method can be used for case $n = m \geq 2$ and $k = 7$, but more details are required. The only open cases for the above conjecture when $n = m$ are $k = 4, 5, 6$. Here, we investigate the case $k = 4$, and we show that the conjecture holds for $k = 4$ when $n > m$ or $n = m$ is odd. When $n = m$ is even, we show that $R(\mathcal{C}_n^4, \mathcal{C}_n^4)$ is between two values with difference one.

1. Introduction

For given k-uniform hypergraphs $\mathcal{G}$ and $\mathcal{H}$, the *Ramsey number* $R(\mathcal{G}, \mathcal{H})$ is the smallest positive integer N such that in every red–blue coloring of the edges of the complete k-uniform hypergraph $\mathcal{K}_N^k$, there is a red copy of $\mathcal{G}$ or a blue copy of $\mathcal{H}$. A *k-uniform loose cycle* $\mathcal{C}_n^k$ (shortly, a *cycle of length n*) is a hypergraph with vertex set $\{v_1, v_2, \ldots, v_{n(k-1)}\}$ and with the set of n edges $e_i = \{v_{(i-1)(k-1)+1}, v_{(i-1)(k-1)+2}, \ldots, v_{(i-1)(k-1)+k}\}$, $1 \leq i \leq n$, where we use mod $n(k-1)$ arithmetic. Similarly, a *k-uniform loose path* $\mathcal{P}_n^k$ (shortly, a *path of length n*) is a hypergraph with vertex set $\{v_1, v_2, \ldots, v_{n(k-1)+1}\}$ and with the set of n edges $e_i = \{v_{(i-1)(k-1)+1}, v_{(i-1)(k-1)+2}, \ldots, v_{(i-1)(k-1)+k}\}$, $1 \leq i \leq n$. For an edge $e_i = \{v_{(i-1)(k-1)+1}, v_{(i-1)(k-1)+2}, \ldots, v_{i(k-1)+1}\}$ of a given loose path (also a given loose cycle) $\mathcal{K}$, the first vertex ($v_{(i-1)(k-1)+1}$) and the last vertex ($v_{i(k-1)+1}$) are denoted by $f_{\mathcal{K},e_i}$ and $l_{\mathcal{K},e_i}$, respectively. In this paper, we consider the problem of finding the 2-color Ramsey number of 4-uniform loose paths and cycles.

The investigation of the Ramsey numbers of hypergraph loose cycles was initiated by Haxell et al. in [3]. They proved $R(\mathcal{C}_n^3, \mathcal{C}_n^3)$ is asymptotically $\frac{5}{2}n$. This result was extended by Gyárfás, Sárközy and Szemerédi [2] to k-uniform loose cycles. More precisely, they proved that for all $\eta > 0$ there exists $n_0 = n_0(\eta)$ such that for every $n > n_0$, every 2-coloring of $\mathcal{K}_N^k$ with $N = (1+\eta)\frac{1}{2}(2k-1)n$ contains a monochromatic copy of $\mathcal{C}_n^k$.

In [1], Gyárfás and Raeisi determined the value of the Ramsey number of a k-uniform loose triangle and quadrangle. Recently, we proved the following general result on the Ramsey numbers of loose paths and loose cycles in 3-uniform hypergraphs.

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http://dx.doi.org/10.1016/j.dam.2017.04.046
0166-218X/

Theorem 1.1 (*[4]*). *For every $n \geq m \geq 3$,*

$$R(\mathcal{P}^3_n, \mathcal{P}^3_m) = R(\mathcal{P}^3_n, \mathcal{C}^3_m) = R(\mathcal{C}^3_n, \mathcal{C}^3_m) + 1 = 2n + \left\lfloor \frac{m+1}{2} \right\rfloor.$$

In [5], we presented another proof of Theorem 1.1 and posed the following conjecture.

Conjecture 1. *Let $k \geq 3$ be an integer number. For every $n \geq m \geq 3$,*

$$R(\mathcal{P}^k_n, \mathcal{P}^k_m) = R(\mathcal{P}^k_n, \mathcal{C}^k_m) = R(\mathcal{C}^k_n, \mathcal{C}^k_m) + 1 = (k-1)n + \left\lfloor \frac{m+1}{2} \right\rfloor.$$

Also, the following theorem is obtained on the Ramsey number of loose paths and cycles in k-uniform hypergraphs [5].

Theorem 1.2 (*[5]*). *Let $n \geq m \geq 2$ be given integers and $R(\mathcal{C}^k_n, \mathcal{C}^k_m) = (k-1)n + \lfloor \frac{m-1}{2} \rfloor$. Then, $R(\mathcal{P}^k_n, \mathcal{C}^k_m) = (k-1)n + \lfloor \frac{m+1}{2} \rfloor$ and $R(\mathcal{P}^k_n, \mathcal{P}^k_{m-1}) = (k-1)n + \lfloor \frac{m}{2} \rfloor$. Moreover, for $n = m$ we have $R(\mathcal{P}^k_n, \mathcal{P}^k_m) = (k-1)n + \lfloor \frac{m+1}{2} \rfloor$.*

Using Theorem 1.2, one can easily see that Conjecture 1 is equivalent to the following.

Conjecture 2. *Let $k \geq 3$ be an integer number. For every $n \geq m \geq 3$,*

$$R(\mathcal{C}^k_n, \mathcal{C}^k_m) = (k-1)n + \left\lfloor \frac{m-1}{2} \right\rfloor.$$

Recently, it is shown that Conjecture 2 holds for $n = m$ and $k \geq 8$ (see [6]). As we mentioned in [6], our methods can be used to prove Conjecture 2 for $n = m$ and $k \geq 7$. Therefore, based on Theorem 1.1, the cases $k = 4, 5, 6$ are the only open cases for Conjecture 2 when $n = m$ (the problem of determines the diagonal Ramsey number of loose cycles). In this paper, we extend the method that used in [5] and show that Conjecture 2 holds for $k = 4$, unless $n = m$ and n is even. In this case, we show that $R(\mathcal{C}^4_n, \mathcal{C}^4_n)$ is between two values with difference one. More precisely, we show the following theorem.

Theorem 1.3. *For every $n \geq m + 1 \geq 4$,*

$$R(\mathcal{C}^4_n, \mathcal{C}^4_m) = 3n + \left\lfloor \frac{m-1}{2} \right\rfloor.$$

Moreover, if n is odd, then $R(\mathcal{C}^4_n, \mathcal{C}^4_n) = 3n + \left\lfloor \frac{n-1}{2} \right\rfloor$. Otherwise,

$$3n + \left\lfloor \frac{n-1}{2} \right\rfloor \leq R(\mathcal{C}^4_n, \mathcal{C}^4_n) \leq 3n + \left\lfloor \frac{n-1}{2} \right\rfloor + 1.$$

Consequently, using Theorem 1.2, we obtained the values of some Ramsey numbers involving paths. Here, we give a sketch of our proof for Theorem 1.3. We consider a two coloring of $\mathcal{K}^4_{3n+\lfloor \frac{m-1}{2} \rfloor}$ by colors red and blue. Our proof is based on induction on $n+m$ and relies on the following approach: We consider the largest red cycle and show that if this cycle cannot be extended to a red $\mathcal{C}^4_n$, then there are many blue paths of lengths 2 between that cycle and other vertices. Then, we show that we can construct a blue copy of $\mathcal{C}^4_m$ by combining these paths.

Throughout the paper, by Lemma 1 of [1], it suffices to prove only the upper bound for the claimed Ramsey numbers. Throughout the paper, for a 2-edge colored hypergraph $\mathcal{H}$, we denote by $\mathcal{H}_{\text{red}}$ and $\mathcal{H}_{\text{blue}}$ the induced hypergraphs on red edges and blue edges, respectively. Also, we denote by $|\mathcal{H}|$ and $\|\mathcal{H}\|$ the number of vertices and edges of $\mathcal{H}$, respectively.

2. Preliminaries

In this section, we prove some lemmas that will be needed in our main results. Also, we recall some results from [1] and [5].

Theorem 2.1 (*[1]*). *For every $k \geq 3$,*

(a) $R(\mathcal{P}^k_3, \mathcal{P}^k_3) = R(\mathcal{C}^k_3, \mathcal{P}^k_3) = R(\mathcal{C}^k_3, \mathcal{C}^k_3) + 1 = 3k - 1$,

(b) $R(\mathcal{P}^k_4, \mathcal{P}^k_4) = R(\mathcal{C}^k_4, \mathcal{P}^k_4) = R(\mathcal{C}^k_4, \mathcal{C}^k_4) + 1 = 4k - 2$.

Theorem 2.2 (*[5]*). *Let $n, k \geq 3$ be integer numbers. Then,*

$$R(\mathcal{C}^k_3, \mathcal{C}^k_n) = (k-1)n + 1.$$

In order to state our main results, we need some definitions. Let $\mathcal{H}$ be a 2-edge colored complete 4-uniform hypergraph, $\mathcal{P}$ be a loose path in $\mathcal{H}$ and W be a set of vertices with $W \cap V(\mathcal{P}) = \emptyset$. By a *$\varpi_S$-configuration*, we mean a copy of $\mathcal{P}^4_2$ with edges

$$\{x, a_1, a_2, a_3\}, \{a_3, a_4, a_5, y\},$$

Please cite this article in press as: G.R. Omidi, M. Shahsiah, Ramsey numbers of 4-uniform loose cycles, Discrete Applied Mathematics (2017), http://dx.doi.org/10.1016/j.dam.2017.04.046.