ARTICLE IN PRESS

Discrete Applied Mathematics ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Bondage number of the strong product of two trees☆

Weisheng Zhao [a], Fan Wang [b], Xiaolu Gao [c], Hao Li [d,*]

[a] Institute for Interdisciplinary Research, Jianghan University, Wuhan, Hubei 430056, China
[b] School of Sciences, Nanchang University, Nanchang, Jiangxi 330000, China
[c] School of Mathematics and Statistics, Lanzhou University, Lanzhou, Gansu 730000, China
[d] Laboratoire de Recherche en Informatique, UMR 8623, C.N.R.S.-Université Paris-sud, F-91405, Orsay, France

ARTICLE INFO

Article history:
Received 2 November 2016
Received in revised form 16 June 2017
Accepted 22 June 2017
Available online xxxx

Keywords:
Bondage number
Strong product
Trees

ABSTRACT

The bondage number $b(G)$ of a nonempty graph G is the cardinality of a minimum set of edges whose removal from G results in a graph with domination number greater than that of G. It is known that $b(T) \leq 2$ for any nontrivial tree T. In this paper, we obtain that the bondage number of the strong product of two nontrivial trees $b(T \boxtimes T')$ is equal to $b(T)b(T')$ or $b(T)b(T') + 1$, which implies that $b(T \boxtimes T')$ is equal to 1, 2, 3, 4 or 5.

1. Introduction

1.1. Some basic definitions

The graphs considered in this paper are finite, undirected and simple. Let G be a graph with vertex-set $V(G)$ and edge-set $E(G)$. As usual, we use $N_G(v)$ and $N_G[v]$ to denote the open and closed neighborhood of a vertex v in G, respectively. A subset D of $V(G)$ is called a *dominating set* of G if every vertex of G is either in D or adjacent to a vertex of D. The *domination number* $\gamma(G)$ is the cardinality of a minimum dominating set of G. For simplicity, we denote by $\underline{MDS}(G)$ the set of all the minimum dominating sets G. That is, $\underline{MDS}(G) = \{D \mid D$ is a minimum dominating set composed of $G\}$. A subset S of $E(G)$ is called a *bondage edge set* of a nonempty graph G if $\gamma(G - S) > \gamma(G)$. In particular, an edge e is called a *bondage-edge* of G if $\gamma(G - e) > \gamma(G)$. The *bondage number* of G, denoted by $b(G)$, is the cardinality of a minimum bondage edge set of G.

Let G and H be two graphs. The *strong product* (see [6]) of G and H, denoted by $G \boxtimes H$, is a graph such that $V(G \boxtimes H) = V(G) \times V(H)$, two vertices (g_1, h_1) and (g_2, h_2) are adjacent if and only if either $g_1 = g_2$ and $h_1h_2 \in E(H)$, or $g_1g_2 \in E(G)$ and $h_1 = h_2$, or $g_1g_2 \in E(G)$ and $h_1h_2 \in E(H)$ (refer to Fig. 1.1).

Definition 1.1 (*[14]*). (1) A vertex $v \in V(G)$ is called *universal* if v belongs to every minimum dominating set of G.

(2) A vertex $v \in V(G)$ is called *idle* if v does not belong to any minimum dominating set of G.

(3) A vertex $v \in V(G)$ is called *alterable* if v is neither universal nor idle.

☆ This work is supported by NSFC (Grant Nos. 61501208, 11501282 & 11571155); Scientific Research Foundation of Jianghan University (No. 1019-06240001).

* Corresponding author.
E-mail addresses: weishengzhao101@aliyun.com (W. Zhao), wangfan@ncu.edu.cn (F. Wang), gaoxl14@lzu.edu.cn (X. Gao), Hao.Li@lri.fr (H. Li).

http://dx.doi.org/10.1016/j.dam.2017.06.019
0166-218X/

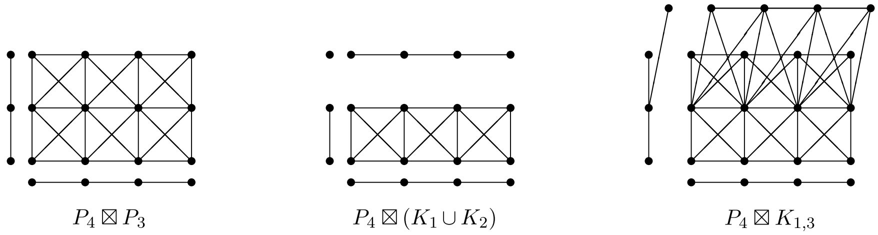

Fig. 1.1. Examples of strong product.

Definition 1.2. [14] Let e be a bondage-edge of G.
(1) e is called a *genuine* bondage-edge if e is incident with two alterable vertices of G.
(2) e is called a *nongenuine* bondage-edge if e is incident with one universal vertex and one idle vertex of G.
There is no other kind of bondage-edge.

Definition 1.3. A vertex $v \in V(G)$ is called *critical* if and only if $\gamma(G-v) \le \gamma(G)-1$. (Note that $\gamma(G-v) \le \gamma(G)-1 \Leftrightarrow \gamma(G-v) = \gamma(G)-1$.) In particular, we agree that the single vertex of a trivial graph is critical.

Definition 1.4. An edge incident with a vertex of degree one in G is called a *pendant edge* of G.

1.2. Background and the main result of this paper

The bondage number of a graph was coined by Fink, Jacobson, Kinch and Roberts [5], which is a parameter to measure the vulnerability of a communication network under link failure. Hu and Xu [9] showed that the problem of determining the bondage number of a general graph is NP-hard. For a thorough study on the advances of bondage, readers are suggested to refer to a survey [15].

Product graphs are one kind of important networks. A large network can be constructed by the product of several small graphs. On the study of the bondage number of the product graphs, the authors mainly focus on the Cartesian product graphs. For example, $b(K_n \Box K_n)$ for $n \ge 3$ [7,13], $b(C_n \Box P_2)$ for $n \ge 3$ [4], $b(C_n \Box C_3)$ for $n \ge 4$ [12], $b(C_n \Box C_4)$ for $n \ge 4$ [10], $b(C_n \Box C_5)$ for $n \not\equiv 3 \pmod 5$ and $n \ge 5$ [2], $b(P_n \Box P_2)$, $b(P_n \Box P_3)$ and $b(P_n \Box P_4)$ for $n \ge 2$ [8] have been determined.

Recently, the authors begin to study the bondage number of the strong product graphs. Dettlaff, Lemańska and Yero [3] obtained the value of $b(P_m \boxtimes P_n)$ for $m, n \not\equiv 1 \pmod 3$. Zhao and Zhang [19] determined the complete result of the value of $b(P_m \boxtimes P_n)$ by using a different method. They also got the value of $b(K_m \boxtimes P_n)$ and some upper bounds of the strong product of a graph and a tree under different conditions in [20] and [18], respectively.

In this paper, we obtain the bondage number of the strong product of two trees. Before our introduction, we need to define some sets of nontrivial trees. For the bondage of trees, we have the following two theorems.

Proposition 1.5 ([14]). *Let T be a nontrivial tree. If u is universal, then there exists a nongenuine bondage-edge uv of T.*

Proposition 1.6 ([1,5]). *If T is a nontrivial tree, then $b(T) \le 2$.*

From Propositions 1.5 and 1.6, we know that: for a nontrivial tree T, $b(T)$ is equal to 1 or 2, and $b(T)=1$ if T has a universal vertex. Let $\mathscr{T}_1$ and $\mathscr{T}_2$ denote the sets of nontrivial trees with bondage numbers 1 and 2, respectively. We partition $\mathscr{T}_1$ into two classes: $\mathscr{T}_1^a$ and $\mathscr{T}_1^b$, and separate a subclass $\mathscr{T}_1^{a_0}$ from $\mathscr{T}_1^a$. We also partition $\mathscr{T}_2$ into two classes: $\mathscr{T}_2^\alpha$ and $\mathscr{T}_2^\beta$, and separate a subclass $\mathscr{T}_2^{\alpha_0}$ from $\mathscr{T}_2^\alpha$. The definitions of $\mathscr{T}_1^a$, $\mathscr{T}_1^{a_0}$, $\mathscr{T}_1^b$, $\mathscr{T}_2^\alpha$, $\mathscr{T}_2^{\alpha_0}$ and $\mathscr{T}_2^\beta$ are listed as follows. (Remark: (i) the symbol $P(T)$ is defined as the set composed of all the pendant edges of T; (ii) $s_0t_0 \notin P(T)$ means that $s_0t_0 \notin E(T)$ or $s_0t_0 \in E(T)-P(T)$.) The examples of these classes of trees are shown in Fig. 1.2, where we use "×" to label the bondage-edges and use black, gray and white vertices to represent the universal, alterable and idle vertices, respectively.

$\mathscr{T}_1^a = \{T \in \mathscr{T}_1 \mid T \text{ has a universal vertex}\}$;

$\mathscr{T}_1^{a_0} = \{T \in \mathscr{T}_1 \mid T \text{ has a vertex } u \text{ such that all the elements of } N_T(u) \text{ are idle in } T\}$;

$\mathscr{T}_1^b = \{T \in \mathscr{T}_1 \mid T \text{ has no universal vertex}\}$;

$\mathscr{T}_2^\alpha = \{T \in \mathscr{T}_2 \mid \text{there exist } s_0, t_0 \in V(T) \text{ such that } s_0t_0 \notin P(T) \text{ and } \{s_0, t_0\} \cap D \ne \emptyset \text{ for any } D \in \underline{MDS}(T)\}$;

$\mathscr{T}_2^{\alpha_0} = \{T \in \mathscr{T}_2 \mid \text{there exist } s_0, t_0 \in V(T) \text{ such that } s_0t_0 \notin P(T), \text{ and } N_T(s_0) \cap D = \emptyset \text{ or } N_T(t_0) \cap D = \emptyset \text{ for any } D \in \underline{MDS}(T)\}$;

$\mathscr{T}_2^\beta = \{T \in \mathscr{T}_2 \mid \text{for any } s, t \in V(T) \text{ with } st \notin P(T), \text{ there exists } D_0 \in \underline{MDS}(T) \text{ such that } \{s, t\} \cap D_0 = \emptyset\}$.

Please cite this article in press as: W. Zhao, et al., Bondage number of the strong product of two trees, Discrete Applied Mathematics (2017), http://dx.doi.org/10.1016/j.dam.2017.06.019.