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Axiomatic characterization of the center function. The case of universal axioms

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ABSTRACT

The center function on a connected graph G has as input a sequence of vertices of G. The output is the set of vertices that minimize the maximum distance to the entries of the input. If the input is a sequence containing each vertex of G once, then the output is just the classical center of G. This paper studies the center function from the viewpoint of consensus theory. We present consensus axioms that are satisfied by the center function on all connected graphs. Next, we study classes of graphs on which the center function is characterized by such 'universal' axioms only. We present two instances: the graphs with a dominating vertex (that is, a vertex adjacent to all other vertices), and the paths. Trees in general do not fall into this category. But we show that trees with diameter at most five have such a characterization. Our approach is to highlight unexpected analogies between the center function and the median function.

1. Introduction

In 1869, Camille Jordan published his paper [17] on central points in a tree-like structure of curves in a plane. In modern graph-theoretic language, he determined the center and the median (centroid) of a tree. The aim of his paper was to determine what was fixed in a tree under every automorphism. So his interest was from the view point of emerging group theory. Nowadays, the center and the median are the prime examples of finding an optimal location with respect to certain distance criteria. Given the locations of a set of clients in a graph or network, the center consists of the vertices that minimize the maximum distance to the clients. The median consists of the vertices that minimize the average distance to the clients, which is equivalent to minimizing the distance sum to the clients.

Such location problems can also be phrased as consensus problems. The origins of consensus theory date back to 1951, when Arrow's seminal paper [1] appeared, see also [2,3,5,9,25]. In a consensus problem, the clients want to reach consensus in a rational way. This is modeled by a *consensus function*. The input is the location of the clients, the output is the set of locations they agree upon. Thus, we can get the center function and the median function. Rationality of the process is guaranteed by certain rules, called *consensus axioms*, that are imposed on the consensus function. One of the aims in

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consensus theory is to find consensus axioms that actually determine the consensus function. Otherwise said: given a consensus function L, try to find an axiomatic characterization of L. Here, we want these axioms to be as simple and natural as possible. Of course, it will depend on the consensus function L, and on the graph on which L is defined, how simple and natural these axioms will be.

The median function and the center function are special cases of the so-called ℓ_p-function studied in [22]. This function uses the ℓ_p-norm to measure the distance between a vertex and a profile. As is well known, the ℓ_1-norm measures the sum of the distances to a vertex, which is precisely what the median function does, and the ℓ_∞-norm measures the maximum distance to a vertex, which is precisely what the center function does.

The median function as a consensus function is well-studied, see e.g. [6,13,19,20,23,24,26,31,37]. Other functions that have been studied are the mean function [15,21,37], the ℓ_p-function [22], the antimedian function [4,7], and finally the center function [27,33,35]. Most of these papers deal with the respective function on trees. The antimedian function has been studied on paths and complete graphs minus a matching. The median function is the only function that has been studied systematically on other classes, notably median graphs and median semi-lattices. Our focus in this paper is the center function. So far, the center function has been studied only on trees. McMorris, Roberts and Wang [27] gave a characterization of the center function on trees involving four axioms: population invariance, middleness, quasi-consistency and redundancy. The first three axioms are what we call *universal axioms* for the center function, because it satisfies these axioms on all connected graphs. The redundancy axiom is a class-dependent axiom: for instance, it holds on trees, but not on all connected graphs.

The median function is an interesting example from the viewpoint of this paper. It satisfies three very nice and simple axioms on all connected graphs: anonymity, betweenness and consistency. In [20], the so-called *ABC*-Problem was formulated. One of the questions here is: on which graphs is the median function characterized by these three basic axioms only? In [31], it was shown that this is the case on median graphs. This result depended heavily on the rich structure theory for median graphs, see e.g. [18,28,29]. In [20], some more classes are identified. In this paper, we focus on the analogous question for the center function. First, can we determine universal axioms that hold for the center function on all connected graphs? Second, can we find classes of graphs on which the center function is actually characterized by such universal axioms only?

In Section 2, we explain the case of the median function in more detail, and provide essential information on gated sets. In Section 3, we collect such universal axioms for the center function. In Section 4, we present a first example of a class on which universal axioms characterize the center function. Thus, we extend the analogy between the median case and the center case. In Section 5, we present our second example, viz. the paths. We also show that this characterization holds for all trees with diameter at most 5, but that on trees with larger diameter, we need extra, so-called class-specific axioms. In all cases, we present examples that show that the axioms involved are independent, and thus are all necessary.

2. Preliminaries

Throughout this paper, $G = (V, E)$ is a finite, connected, simple, loopless graph with vertex set V. The *distance* $d(u, v)$ between two vertices u and v in G is the length of a shortest u, v-path, usually called a u, v-geodesic. The *interval* $I_G(u, v)$ between u and v in G is the set of vertices on the u, v-geodesics, that is,

$$I_G(u, v) = \{w \mid d(u, w) + d(w, v) = d(u, v)\}.$$

When no confusion arises, we write $I = I_G$. The function $I: V \times V \to 2^V - \emptyset$ is the *interval function* on G, where $2^V - \emptyset$ is the family of nonempty subsets of V. A first extensive study of I can be found in [28]. A subset S of V is *convex* if $I(u, v) \subseteq S$ for any two vertices u and v in S. Clearly, the empty set as well as V are convex sets. Also, the intersection of any two convex sets is again convex. Hence, the family of convex sets in G forms a *convexity* in the sense of abstract convexity theory, see [36]. The *convex closure* $Con(W)$ of a subset W of V is the intersection of all convex sets containing W, that is, the smallest convex set containing W.

A *profile* of *length* k on V is a nonempty sequence $\pi = (x_1, x_2, \ldots, x_k)$. Note that vertices may occur more than once in a profile. We denote the length of π by $k = |\pi|$. We call $x_1, x_2, \ldots, x_k$ the *entries* of π. A *vertex* of π is a vertex that occurs as an entry in π. The *carrier set* $\{\pi\}$ of π is the set of vertices that occur in π. So, if a vertex occurs more than once as an entry in π, then we have $|\{\pi\}| < |\pi|$. We write $Con(\{\pi\}) = Con(\pi)$. A *single-occurrence profile* is a profile π in which each vertex occurs at most once, that is, $|\pi| = |\{\pi\}|$. Let $\rho = (x_1, x_2, \ldots, x_k)$ and $\sigma = (y_1, y_2, \ldots, y_\ell)$ be two profiles. The profile $(x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_\ell)$ is called a *concatenation* of ρ and σ, and is denoted by $\rho\sigma$. Note that in most cases we have $\rho\sigma \neq \sigma\rho$. For any entry x_i in π, we write the profile obtained by deleting this entry as $\pi - x_i$. Note that $\pi - x_i$ has length $k - 1$. Of course this has only meaning for $k \geq 2$. We call $\pi - x_i$ an *entry-deleted profile*. Note that we do not delete other occurrences of this vertex.

We denote by V^* the set of all profiles of finite length on V. A *consensus function* on a graph $G = (V, E)$ is a function $L: V^* \to 2^V - \emptyset$. For convenience, we write $L(x_1, x_2, \ldots, x_k)$ instead of $L((x_1, x_2, \ldots, x_k))$, for any profile $\pi = (x_1, x_2, \ldots, x_k)$.

Let $\pi = (x_1, x_2, \ldots, x_k)$ be a profile on G. For a vertex v in G, we define the *maximum distance* from v to π by $R(v, \pi) = \max\{d(v, x_i) \mid 1 \leq i \leq k\}$. A vertex x minimizing this maximum distance is called a *center* of π. The *center function Cen* on G is the consensus function given by

$$Cen(\pi) = \{v \mid v \text{ is a center of } \pi\}.$$

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