Discrete Applied Mathematics 227 (2017) 84–94

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

On the bandwidth of the Kneser graph

Tao Jiang [a], Zevi Miller [a,*], Derrek Yager [b]

[a] Department of Mathematics, Miami University, Oxford, OH 45056, USA
[b] Department of Mathematics, University of Illinois, Champaign-Urbana, IL, USA

ARTICLE INFO

Article history:
Received 30 November 2015
Received in revised form 23 November 2016
Accepted 12 April 2017
Available online 29 May 2017

Keywords:
Bandwidth
Kneser graph

ABSTRACT

Let $G = (V, E)$ be a graph on n vertices and $f : V \to [1, n]$ a one to one map of V onto the integers 1 through n. Let *dilation*$(f) = \max\{|f(v) - f(w)| : vw \in E\}$. Define the *bandwidth* $B(G)$ of G to be the minimum possible value of *dilation*(f) over all such one to one maps f. Next define the *Kneser Graph* $K(n, r)$ to be the graph with vertex set $\binom{[n]}{r}$, the collection of r-subsets of an n element set, and edge set $E = \{AB : A, B \in \binom{[n]}{r}, A \cap B = \emptyset\}$. For fixed $r \geq 4$ and n growing we show that

$$B(K(n, r)) = \binom{n-1}{r} + \frac{1}{2}\binom{n-4}{r-1} - \frac{1}{2}\binom{n-1}{r-2} + O(n^{r-4}).$$

1. Introduction

We begin with some notation. Let $[n] = \{1, 2, 3, \ldots, n\}$, which we view as our canonical set of size n. For any finite set S we let $\binom{S}{r}$ be the collection of r-subsets of S. In particular $\binom{[n]}{r}$ will be the collection of r-subsets of $[n]$. For integers $a < b$ we let $[a, b]$ denote the set of integers x satisfying $a \leq x \leq b$.

Now let $\mathcal{A}$ and $\mathcal{B}$ be two families of subsets of $[n]$. We say $\mathcal{A}$ is *intersecting* if $A_1 \cap A_2 \neq \emptyset$ for all pairs $A_1, A_2 \in \mathcal{A}$. Further $\mathcal{A}$ is *nontrivial* if $\cap_{A \in \mathcal{A}} A = \emptyset$, and is *trivial* otherwise. The pair of families $\mathcal{A}, \mathcal{B}$ is *cross intersecting* if $A \cap B \neq \emptyset$ for all pairs of sets A, B, where $A \in \mathcal{A}$ and $B \in \mathcal{B}$. A *matching* of $\mathcal{A}$ is a collection of sets in $\mathcal{A}$ that are pairwise disjoint. For $S \subset [n]$ we let $V(S) = \{x \in [n] : x \in S\}$, and we let $V(\mathcal{A}) = \cup_{S \in \mathcal{A}} V(S)$ (the vertex set of $\mathcal{A}$). We sometimes refer to the sets in $\mathcal{A}$ as *members* of $\mathcal{A}$,. In the sections which follow the introduction, we use small case latin letters $x, y, u, v, \ldots$ to stand for elements of $[n]$, capital letter $A, B, \ldots$ to stand for subsets of size at least 2 from $[n]$ (mostly these will be r-sets), and calligraphy $\mathcal{A}, \mathcal{B}, \ldots$ to stand for collection (or families) of subsets of $[n]$. Apart from these conventions, we use standard graph theoretic or combinatorial notation, as may be found for example in [47]. Additional notation will be defined where it is initially used in the text.

Now define the *Kneser Graph* $K(n, r)$ to be the graph with vertex set $V = \binom{[n]}{r}$, and edge set $E = \{AB : A, B \in \binom{[n]}{r}, A \cap B = \emptyset\}$. We can suppose that $n \geq 2r$ since otherwise $K(n, r)$ has no edges. Clearly $K(n, r)$ has $\binom{n}{r}$ vertices, is regular of degree $\binom{n-r}{r}$, and it can be shown that it is both vertex and edge transitive [39]. The Kneser Graph arises in several examples; $K(n, 1)$ is just the complete graph K_n on n vertices, $K(n, 2)$ is the complement of the line graph of K_n, $K(2n - 1, n - 1)$ is also known as the odd graph O_n, and $K(5, 2)$ is isomorphic to the Petersen graph. The diameter of $K(n, r)$ was shown to be $\lceil \frac{r-1}{n-2r} \rceil + 1$ in [45], and $K(n, r)$ was shown to be Hamiltonian for $n \geq \frac{1}{2}(3r + 1 + \sqrt{5r^2 - 2r + 1})$ in [7].

A longstanding problem on $K(n, r)$ was Kneser's conjecture; that the chromatic number satisfies $\chi(K(n, r)) = n - 2r + 2$ if $n \geq 2r$ and of course $\chi(K(n, r)) = 1$ otherwise. The upper bound is achieved by a simple coloring; color an r-set by its

* Corresponding author.
E-mail addresses: jiangt@miamioh.edu (T. Jiang), millerz@miamioh.edu (Z. Miller), yager2@illinois.edu (D. Yager).

http://dx.doi.org/10.1016/j.dam.2017.04.008

largest element if this element is at least $2r$, and otherwise color it by 1. The difficulty was in proving the corresponding lower bound, and this result was first proved by Lovasz [33] using methods of algebraic topology. More elementary, but still topological, proofs were given by Bárány [3] soon after, and by Dol'nikov [13] and Greene [25] later. A mostly combinatorial proof (still with topological elements) was given by Matoušek [36].

Recently some results on a graph labeling problem relating to $K(n, r)$ appeared in the literature [30]. Let $G = (V, E)$ be a graph on n vertices and $f : V \to C_n$ a one to one map of the vertices of G to the cycle C_n on n vertices. Let $|f| = \min\{dist_{C_n}(f(u), f(v)) : uv \in E\}$, where $dist_{C_n}$ denotes the distance function on C_n; that is, $dist_{C_n}(x, y)$ is the mod n distance between x and y when we view the vertices of C_n as the integers mod n. Now let $s(G) = \max\{|f|\}$, where the maximum is taken over all such one to one maps f. It is shown in [30] that $s(K(n, 2)) = 3$ when $n \geq 6$, that $s(K(n, 3)) = 2n - 7$ or $2n - 8$ for n sufficiently large, and that for fixed $r \geq 4$ and n sufficiently large we have $\frac{2n^{r-2}}{(r-2)!} - \frac{(\frac{7}{2}r-2)n^{r-3}}{(r-3)!} - O(n^{r-4}) \leq s(K(n, r)) \leq \frac{2n^{r-2}}{(r-2)!} - \frac{(\frac{7}{2}r-3.2)n^{r-3}}{(r-3)!} + o(n^{r-3})$.

This paper considers the following related well known graph labeling problem. Let $G = (V, E)$ be a graph on n vertices. Now consider $f : V \to [1, n]$ a one to one map, and let $dilation(f) = \max\{|f(v) - f(w)| : vw \in E\}$. Define the *bandwidth* $B(G)$ of G to be the minimum possible value of $dilation(f)$ over all labelings f.

There is an extensive literature on the bandwidth of graphs and related labeling problems (see [8] and [12] for surveys). Apart from its intrinsic interest as a combinatorial problem, bandwidth has connections to other areas in pure and applied mathematics. Its relevance to Ramsey theory and extremal problems was shown in [1,41,6], and [5].

In applied directions, the study of graph bandwidth was motivated by matrix problems in numerical analysis. Here consider a symmetric nxn and $0 - 1$ matrix M. Define $\beta(M)$ to be the minimum integer b such that every nonzero entry of M is located in the set of entries $\{M_{ij} : i - b \leq j \leq i + b, 1 \leq i \leq n\}$ of M. Now given a permutation π of $[n]$, consider the matrix M_π obtained from M by applying π simultaneously to the columns and to the rows of M. The minimum of $\beta(M_\pi)$ over all permutations π is called the bandwidth $B(M)$ of M. Consider the graph G whose adjacency matrix is the original M, making the assumption that the diagonal entries of M are 0. Then it is straightforward to see that $B(G)$ is the same as $B(M)$, where the labelings of G correspond to the permutations π applied to the rows and columns of M. Now the interest in $B(M)$ arises because certain operations on matrices (like Choleski factorization of nonsingular matrices, see [43]) require less space and can be speeded up when the bandwidth of the matrix is small. More recent applications of bandwidth have appeared in the context of information retrieval in browse hypertext (see [44]).

There are general upper and lower bounds as well as exact formulas for $B(G)$ for certain graph classes in terms of natural graph parameters like maximum degree and diameter among others [8] and [12]. We will return to these shortly in the context of the Kneser graph. For example, exact formulas are known for $B(G)$ when G is a path, cycle, complete graph, complete multipartite graph, complete k-ary tree, grid, or hypercube (again, see the above surveys); the formula in the last three cases involving nontrivial arguments (see [26] for hypercubes and [9] for grids). There are also the bounds $B(T) \leq \frac{5n}{\log_\Delta(n)}$ when T is a tree [8], and more generally $B(G) \leq \frac{20n}{\log_\Delta(n)}$ when G is a planar graph [5], both having n vertices and bounded maximum degree Δ.

Concerning the algorithmic complexity of the graph bandwidth problem, it was shown that this problem (suitably stated as a decision problem) is NP-complete [42], even when restricted to the class of trees of maximum degree 3 [23]. Let $B^*(G)$ to be the *topological bandwidth* of G. This is defined as the minimum of $B(H)$ over all graphs H which are refinements of G; that is, those H which can be obtained from G by inserting an arbitrary number of points of degree 2 along any of the edges of G. It was shown in [35] that calculating $B^*(G)$ is NP-complete. But in contrast to the NP-completeness of bandwidth for the class of trees of maximum degree three, it was shown independently in [35] and [38] that topological bandwidth is polynomial time solvable for this class of trees, in fact in time $O(nlog(n))$ in the former paper and in time $O(n)$ in the latter paper, where n is the number of vertices in a tree of maximum degree three. Next define a "caterpillar" to be a tree for which the removal of all leaves results in a path P. Any refinement C^* of C must then be an edge disjoint union of P^*, the refinement of P in C^*, together with a collection of "hairs" of C^*. Each hair h is a refinement in C^* of a length 1 path in C joining a point on P to a leaf of C. In [37] it was shown that $B(C)$ is polynomial time computable when C is a caterpillar, and independently in [2] it was shown that $B(C^*)$ is polynomial time computable when C^* is a caterpillar refinement with all hairs having length 1 or 2. To cap off the previous two results it was shown in [40] that the bandwidth problem for refinements of caterpillars with hair length at most 3 is NP-complete.

Concerning approximation algorithms, it was shown in [11] that the problem of approximating bandwidth on arbitrary graphs to within a constant factor is NP-complete, even when we restrict to the class of refinements of caterpillars. Again for this class, in [16] a polynomial time $O(\frac{log(n)}{log(log(n))})$-approximation algorithm is given, and a $(1 + \epsilon)$-approximation algorithm is given which runs in time $2^{O(\sqrt{\frac{n}{\epsilon}})}$. Now let $0 < \delta < 1$, and define a graph G on n vertices to be δ-dense if the minimum degree of G is at least δn, where n is the number of vertices in G. In [32] a randomized algorithm of running time $n^{O(1/\delta)}$ was given which for any δ-dense graph G produces a labeling f of G satisfying $dilation(f) \leq 3B(G)$ with high probability.

Some computational approaches to the bandwidth problem have been proposed. Two older heuristics can be found in [10] and [24]. A recent paper [46] obtains both lower and upper computational bounds for bandwidth in graphs. The lower bound is based on a new lower bound for the minimum cut problem, which the authors obtain by strengthening a known semidefinite programming relaxation of the quadratic assignment problem. The upper bound is a heuristic based on the Cuthill–McKee algorithm [10] and yields improved upper bounds. Computational results are given for the bandwidth of