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Note

On the bipartite graph packing problem

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ABSTRACT

The graph packing problem is a well-known area in graph theory. We consider a bipartite version and give almost tight conditions.

1. Notation

We consider only simple graphs. Throughout the paper we use common graph theory notations: $d_G(v)$ (or briefly, if G is understood from the context, $d(v)$) is the degree of v in G, and $\Delta(G)$ is the maximal and $\delta(G)$ is the minimal degree of G, and $e(X, Y)$ is the number of edges between X and Y for $X \cap Y = \emptyset$. For any function f on V let $f(X) = \sum_{v \in X} f(v)$ for every $X \subseteq V$. $\pi(G)$ is the degree sequence of G.

2. Introduction

Let G and H be two graphs on n vertices. We say that G and H *pack* if and only if K_n contains edge-disjoint copies of G and H as subgraphs.

The graph packing problem can be formulated as an embedding problem, too. G and H pack if and only if H is isomorphic to a subgraph of $\overline{G}$ ($H \subseteq \overline{G}$).

A classical result in this field is the following theorem of Sauer and Spencer.

Theorem 1 (*Sauer, Spencer [14]*). *Let G_1 and G_2 be graphs on n vertices with maximum degrees Δ_1 and Δ_2, respectively. If $\Delta_1 \Delta_2 < \frac{n}{2}$, then G_1 and G_2 pack.*

Many questions in graph theory can be formulated as special packing problems, see [9].

We study the **bipartite packing problem** as it is formulated by Catlin [3], Hajnal and Szegedy [7] and was used by Hajnal for proving deep results in complexity theory of decision trees [6]. We are not going to state what Hajnal and Szegedy proved in [7], since it is technically involved. However, we present two previous results in bipartite packing, as in certain cases our main theorem is stronger than those.

Let $G_1 = (A, B; E_1)$ and $G_2 = (S, T; E_2)$ be bipartite graphs with $|A| = |S| = m$ and $|B| = |T| = n$. Sometimes, we use only $G(A, B)$ if we want to say that G is a bipartite graph with classes A and B. Let $\Delta_A(G_1)$ be the maximal degree of G_1 in A. We use $\Delta_B(G_1)$ similarly.

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The bipartite graphs G_1 and G_2 pack in the bipartite sense (i.e. they have a *bipartite packing*) if there are edge-disjoint copies of G_1 and G_2 in $K_{m,n}$.

The bipartite packing problem can be also formulated as a question of embedding. The bipartite graphs $G_1 = (A, B; E)$ and G_2 pack if and only if G_2 is isomorphic to a subgraph of $\widetilde{G}_1$, which is the bipartite complement of G_1, i.e. $\widetilde{G}_1 = (A, B; (A \times B) - E)$.

First, we present the result of Wojda and Vaderlind. Before formulating their theorem, we need to introduce three families of graph pairs which they use [17].

Let Γ_1 be the family of pairs $\{G(L, R), G'(L', R')\}$ of bipartite graphs that G contains a star (i.e. one vertex in L is connected to all vertices of R), and in $\delta_{L'}(G') \geq 1$.

Let Γ_2 be the family of pairs $\{G(L, R), G'(L', R')\}$ of bipartite graphs such that $L = \{a_1, a_2\}$, and $d_G(a_1) = d_G(a_2) = 2$; and $L' = \{a'_1, a'_2\}$, $d_{G'}(a'_1) = |R| - 1$, $d_{G'}(a'_2) = 0$, finally, $\Delta_R(G) = \Delta_R(G') = 1$.

The family Γ_3 is the pair $\{G, G'\}$, where $G = K_{2,2} \cup \overline{K_{1,1}}$, and G' is a one-factor.

Theorem 2 ([17]). *Let $G = (L, R; E)$ and $G' = (L', R'; E')$ be two bipartite graphs with $|L| = |L'| = p \geq 2$ and $|R| = |R'| = q \geq 2$, such that*

$$e(G) + e(G') \leq +q + \varepsilon(G, G') \tag{1}$$

where $\varepsilon(G, G') = \min\{p - \Delta_R(G), p - \Delta_{R'}(G'), q - \Delta_L(G), q - \Delta_{L'}(G')\}$.

Then G and G' pack unless either

(i) *$\varepsilon(G, G') = 0$ and $\{G, G'\} \in \Gamma_1$, or*
(ii) *$\varepsilon(G, G') = 1$ and $\{G, G'\} \in \Gamma_2 \cup \Gamma_3$.*

Another theorem in this field is by Wang.

Theorem 3 ([15]). *Let $G(A, B)$ and $H(S, T)$ be two C_4-free bipartite graphs of order n with $|A| = |B| = |S| = |T| = n$, and $e(G) + e(H) \leq 2n - 2$. Then there is a packing of G and H in $K_{n+1,n+1}$ (i.e. an edge-disjoint embedding of G and H into $K_{n+1,n+1}$), unless one is a union of vertex-disjoint cycles and the other is a union of two-disjoint stars.*

For more results in this field, we refer the interested reader to the monograph on factor theory of Yu and Liu [18].

The main question of extremal graph theory is that at most how many edges a G graph might have (or what is its minimum degree) if there is an excluded subgraph H.

Let us formulate our main result in the following theorem as an embedding problem.

Theorem 4. *For every $\varepsilon \in (0, \frac{1}{2})$ there is an $n_0 = n_0(\varepsilon)$ such that if $n > n_0$, and $G(A, B)$ and $H(S, T)$ are bipartite graphs with $|A| = |B| = |S| = |T| = n$ and the following conditions hold, then $H \subseteq G$.*

Condition 1: *$d_G(x) > \left(\frac{1}{2} + \varepsilon\right) n$ holds for all $x \in A \cup B$*

Condition 2: *$d_H(x) < \frac{\varepsilon^4}{100} \frac{n}{\log n}$ holds for all $x \in S$,*

Condition 3: *$d_H(y) = 1$ holds for all $y \in T$.*

We prove Theorem 4 in the next section. First we indicate why we have the bounds in Conditions 1 and 2.

The following two examples show that it is necessary to make an assumption on $\delta(G)$ (see Condition 1) and on $\Delta_S(H)$ (see Condition 2).

First, let $G = K_{\frac{n}{2}+1,\frac{n}{2}-1} \cup K_{\frac{n}{2}-1,\frac{n}{2}+1}$. Clearly, G has no perfect matching. This shows that the bound in Condition 1 is close to being best possible.

For the second example, we choose $G = G(n, n, 0.6)$ to be a random bipartite graph. Standard probability reasoning shows that with high probability, G satisfies Condition 1. However, H cannot be embedded into G, where $H(S, T)$ is the following bipartite graph: each vertex in T has degree 1. In S all vertices have degree 0, except $\frac{\log n}{c}$ vertices with degree $\frac{cn}{\log n}$. The graph H cannot be embedded into G, what follows from the example of Komlós et al. [10].

Remark 5. There are graphs which can be packed using Theorem 4, though Theorem 2 does not imply that they pack.

For instance, let $G(A, B)$ and $H(S, T)$ be bipartite graphs with $|A| = |B| = |S| = |T| = n$. Choose H to be a 1-factor, and G to be a graph such that all vertices in A have degree $\left(\frac{1}{2} + \frac{1}{100}\right) n$. This pair of graphs obviously satisfies the conditions of Theorem 4, thus, H can be embedded into G, which means that H can be packed with the bipartite complement of $\widetilde{G}$.

Now, we check the conditions of Theorem 2 for the graphs $\widetilde{G}$ and H. We know that $e(H) = n$, as H is a 1-factor. Furthermore, in $\widetilde{G}$ each vertex in A has degree $\left(\frac{1}{2} - \frac{1}{100}\right) n$, which means that the number of edges is approximately $\frac{n^2}{4}$. As $\varepsilon(H, \widetilde{G}) \leq n$, the condition of Theorem 2 is obviously not satisfied. □

Remark 6. There are graphs which can be packed using Theorem 4, though Theorem 3 does not imply that they pack. Let G be the union of $\frac{n}{3}$ disjoint copies of C_6's and H be a 1-factor. Obviously, H is C_4-free, but the condition of Theorem 3 is not satisfied for G and H, as $e(G) + e(H) = 3n$.

However, our theorem can give an embedding of H into $\widetilde{G}$, as all conditions of Theorem 4 are satisfied with these graphs. This provides a packing of H and G. □

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