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On the bounds for signless Laplacian energy of a graph

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ABSTRACT

For a simple graph G with n-vertices, m edges and having signless Laplacian eigenvalues $q_1, q_2, \ldots, q_n$, the signless Laplacian energy $QE(G)$ of the graph G is defined as $QE(G) = \sum_{i=1}^{n} | q_i - \overline{d} |$, where $\overline{d} = \frac{2m}{n}$ is the average degree of G. In this paper, we obtain the lower and upper bounds for the signless Laplacian energy $QE(G)$ in terms of clique number ω, maximum degree Δ, number of vertices n, first Zagreb index $M_1(G)$ and number of edges m. As an application, we obtain the bounds for the energy of line graph $\mathscr{L}(G)$ of a graph G in terms of various graph parameters. We also obtain a relation between the signless Laplacian energy $QE(G)$ and the incidence energy $IE(G)$.

1. Introduction

Let $G(V, E)$ be a simple graph with n vertices and m edges having vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$. The adjacency matrix $A = (a_{ij})$ of G is a $(0, 1)$-square matrix of order n whose (i, j)-entry is equal to 1, if v_i is adjacent to v_j and equal to 0, otherwise. Let $D(G) = diag(d_1, d_2, \ldots, d_n)$ be the diagonal matrix associated to G, where $d_i = \deg(v_i)$, for all $i = 1, 2, \ldots, n$. The matrices $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$ are respectively called Laplacian and signless Laplacian matrices and their spectrum are respectively called Laplacian spectrum (L-spectrum) and signless Laplacian spectrum (Q-spectrum) of the graph G. Both $L(G)$ and $Q(G)$ are real symmetric and positive semi-definite matrices. We let $0 = \mu_n \leq \mu_{n-1} \leq \cdots \leq \mu_1$ and $0 \leq q_n \leq q_{n-1} \leq \cdots \leq q_1$ to be the L-spectrum and Q-spectrum of G, respectively. It is well known that $\mu_n = 0$ with multiplicity equal to the number of connected components of G and $\mu_{n-1} > 0$ if and only if G is connected [12]. Moreover $\mu_i = q_i$, for all $i = 1, 2, \ldots, n$ if and only if G is bipartite.

The motivation for Laplacian energy comes from graph energy [10,16,24,29]. The Laplacian energy of a graph G as put forward by Gutman and Zhou (see [22]) is defined as

$$LE(G) = \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right|.$$

For its basic properties, including various lower and upper bounds, (see [11,15,14,27] and the references therein). It is easy to see that

$$tr(L(G)) = \sum_{i=1}^{n} \mu_i = 2m = \sum_{i=1}^{n} q_i = tr(Q(G)),$$

where *tr* is the trace.

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In analogy to Laplacian energy, the signless Laplacian energy $QE(G)$ of G is defined as

$$QE(G)=\sum_{i=1}^{n}\left|q_i-\frac{2m}{n}\right|.$$

Let σ, $1\le\sigma\le n-1$, be the number of signless Laplacian eigenvalues greater than or equal to average degree $\overline{d}=\frac{2m}{n}$. We have,

$$QE(G)=\sum_{i=1}^{n}\left|q_i-\frac{2m}{n}\right|=2S_\sigma^+(G)-\frac{4m\sigma}{n},$$

where $S_\sigma^+(G)=\sum_{i=1}^{\sigma}q_i$. It can be easily seen that

$$QE(G)=2S_\sigma^+(G)-\frac{4m\sigma}{n}=\max_{1\le i\le n-1}\left\{2S_i^+-\frac{4mi}{n}\right\}. \tag{1}$$

For its basic properties, including various lower and upper bounds (see [1,19] and the references therein). The line graph $\mathscr{L}(G)$ of the graph G is the graph having vertex set same as the edge set of G and where two vertices are adjacent in $\mathscr{L}(G)$ if the corresponding edges are adjacent in G.

Let $I(G)$ be the vertex-edge incidence matrix of the graph G. For a graph G with vertex set $V(G)=\{v_1,v_2,\dots,v_n\}$ and edge set $E(G)=\{e_1,e_2,\dots,e_m\}$, the (i,j)-entry of $I(G)$ is 1, if v_i is incident with e_j and 0, otherwise. We know [5] that

$$I(G)I(G)^t=Q(G)\quad\text{and}\quad I(G)^tI(G)=2I_m+A(\mathscr{L}(G)) \tag{2}$$

where I_m is the identity matrix of order m and $Q(G)$ is the signless Laplacian matrix of G. Since the matrices XX^t and X^tX have the same non-zero eigenvalues for any matrix X, it follows that the matrices $Q(G)$ and $2I_m+A(\mathscr{L}(G))$ have the same non-zero eigenvalues.

A clique is the maximum complete subgraph of the graph G. The order of the maximum clique is called clique number of the graph G and is denoted by ω. If H is a subgraph of the graph G, we denote by $G\setminus H$ the graph obtained by removing the edges of H from G. We denote the complete graph and the star on n vertices, respectively, by K_n and $K_{1,n-1}$. For other undefined notations and terminology from spectral graph theory, the readers are referred to [5].

The paper is organized as follows. In Section 2, we give a list of some previously known results. In Sections 3 and 4, we present some lower and upper bounds on signless Laplacian energy $QE(G)$ of graph G in terms of number of vertices n, number of edges m, maximum degree Δ, first Zagreb index $M_1(G)$ and clique number ω of the graph G. These bounds improve some well known lower and upper bounds on $QE(G)$ of G. As application, we obtain the bounds for the energy of line graph $\mathscr{L}(G)$ of a graph G in Section 5. Lastly in Section 6, we obtain a relation between $QE(G)$ and the incidence energy $IE(G)$ of the graph G.

2. Preliminaries

We start with the following observation due to Fulton [13].

Lemma 2.1. *If A and B are two real symmetric matrices of order n, then for any $1\le k\le n$,*

$$\sum_{i=1}^{k}\lambda_i(A+B)\le\sum_{i=1}^{k}\lambda_i(A)+\sum_{i=1}^{k}\lambda_i(B),$$

where $\lambda_i(X)$ is the ith eigenvalue of X.

The next two lemmas can be seen in [31].

Lemma 2.2. *Let G be a graph of order n having maximum degree Δ and largest Q-eigenvalue q_1. Then $q_1\ge\Delta+1$. For a graph G with at least one edge, equality holds if and only if $G\cong K_{1,n-1}$.*

Lemma 2.3. *Let $G'=G+e$ be the graph obtained from G by adding a new edge e. Then the signless Laplacian eigenvalues of G interlace the signless Laplacian eigenvalues of G', that is,*

$$q_1(G')\ge q_1(G)\ge q_2(G')\ge q_2(G)\ge\cdots\ge q_n(G')\ge q_n(G)\ge 0.$$

Let Δ and δ respectively be the maximum and minimum degrees of the graph G, and let $\beta=\frac{1}{2}(\Delta+\delta+\sqrt{(\Delta-\delta)^2+4\Delta})$. The following observation can be found in [4].

Lemma 2.4. *If G is a connected graph of order $n\ge 3$, then $q_1(G)\ge\beta$, with equality if and only if $G\cong K_{1,n-1}$.*

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