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# Structural and spectral properties of corona graphs

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ABSTRACT

Product graphs have been gainfully used in literature to generate mathematical models of complex networks which inherit properties of real networks. Realizing the duplication phenomena imbibed in the definition of corona product of two graphs, we define corona graphs. Given a small simple connected graph which we call seed graph, corona graphs are defined by taking corona product of the seed graph iteratively. We show that the cumulative degree distribution of corona graphs decay exponentially when the seed graph is regular and the cumulative betweenness distribution follows power law when the seed graph is a clique. We determine spectra and signless Laplacian spectra of corona graphs in terms of the corresponding spectra of the seed graph when the seed graph is regular or a complete bipartite graph. Laplacian spectra of corona graphs corresponding to any seed graph is obtained in terms of the Laplacian spectra of the seed graph.



## 1. Introduction

Network modelling using product graphs is an interesting technique to generate complex networks which possibly can capture the properties of real world networks, for example, Kronecker graphs (see [28]). In [28], Leskovec et al. have developed both deterministic and stochastic models by using the Kronecker product of graphs to generate complex networks which inherit properties of real world networks. In [35], the authors have presented generalized graph products methodology to generate different types of complex networks. In this paper, we propose a model for complex networks by using the concept of corona product of graphs.

Corona product of graphs was introduced by Frucht and Harary in 1970 [14]. Given two graphs $G$ and $H$, the corona product of $G$ and $H$ is a graph, we denote it by $G \circ H$, which is constructed by taking $n$ instances of $H$ and each such $H$ gets connected to each node of $G$, where $n$ is the number of nodes of $G$. Starting with a connected simple graph $G$, we define corona graphs which are obtained by taking corona product of $G$ with itself iteratively. In this case, $G$ is called the seed graph for the corona graphs. Thus, given a seed graph, as the number of iterations increases for the formation of corona graphs, it provides a mathematical model of a growing network, in which, as we show in Section 2, the number of nodes grows exponentially in each iteration. In order to discover the structural properties of corona graphs, we focus on diameter, average degree, cumulative degree and betweenness distributions.

Analysis and derivation of spectra, Laplacian spectra and signless Laplacian spectra of corona product of two graphs have received a huge attention in the last decade, for example see [3,4,9,30]. Several attempts have been successfully made to provide computable expressions of eigenvalues and the corresponding eigenvectors of corona product of two simple

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connected graphs. In [4], Barik et al. have determined the spectra of the corona product of two graphs, say $G$ and $H$ when $H$ is a regular graph. They have also determined the Laplacian spectra of the corona product of any two connected graphs. Interestingly, they have provided the expressions of both the eigenvalues and Laplacian eigenvalues of $G \circ H$ in terms of the eigenvalues and Laplacian eigenvalues of $G$, $H$ respectively. In [31], the authors have introduced the concept of coronal of a graph and they have gainfully used it to compute the spectra of $G \circ H$ by using the spectra of $G$ and $H$ and the coronal of $H$ when $H$ is regular or a complete bipartite graph. The formulae of signless Laplacian eigenvalues of $G \circ H$, when $H$ is a complete bipartite graph (not regular) are provided in terms of the signless Laplacian eigenvalues of $G$ and $H$ by introducing the concept of $M$-coronal of a graph in [9]. In [30], the authors have introduced two new notions of corona products as the subdivision-vertex neighbourhood and the subdivision-edge neighbourhood corona of two graphs. They have also determined the adjacency spectra, the Laplacian spectra and the signless Laplacian spectra of these graphs in terms of the corresponding spectra of the constituent graphs.

We would like to emphasize that the use of coronals to derive the spectra and signless Laplacian spectra of $G \circ H$ when $H$ is a complete bipartite graph is not compulsory as we show in this paper by determining the same without using coronal of a graph. In addition to that, one shortcoming of the use of coronal is that, it gives no information about the eigenvectors of the corresponding eigenvalues that may be useful in a variety of applications where the entries of the eigenvectors reveal interesting properties about the nodes and the graph, for example, see [6,28]. In this paper, we calculate the spectra and signless Laplacian spectra of $G \circ H$, when $H$ is a complete bipartite graph, in terms of the corresponding spectra of $G$ and $H$ without using the concept of coronal. As a by-product of our straightforward approach, we could identify the corresponding eigenvectors for both eigenvalues and signless eigenvalues of $G \circ H$. We also investigate the spectra, Laplacian spectra and signless Laplacian spectra of corona graphs generated by different seed graphs. The expressions of these eigenvalues are provided by using iterated functions. This interpretation of eigenvalues provides a global view about the dynamics of these eigenvalues when number of iterations increases in the formation of corona graphs. Finally, the computable expressions of such eigenvalues are provided in appendices.

We mention that corona graphs can be used as models for investigating the duplication mechanism of an individual gene as explained for the formation of new proteins in [23] and the references therein. A similar phenomenon is being observed in corona graphs but instead of a single node, a unit of the seed graph is being duplicated and connected to every node of the existing network. Hence, corona graphs can reveal more insights on duplication phenomena and with the help of proposed graph spectra, the properties of gene duplication and other real world complex networks can be investigated more deeply [35].

Throughout this paper we assume that the seed graph is a simple connected graph. The main contributions of this paper are as follows.

1. We derive the degree sequence, average degree of a node, and the diameter of corona graphs generated by any given seed graph.
2. We show that the cumulative degree distribution of corona graphs decay exponentially when the chosen seed graph is regular, and the cumulative betweenness distribution of corona graphs follows power law when the seed graph is a clique.
3. We provide computable formulae for eigenvalues and signless Laplacian eigenvalues of corona graphs generated by a seed graph which is regular or a complete bipartite graph. Computable expressions of Laplacian eigenvalues of corona graphs are obtained for any seed graph.

We organize the paper as follows. In Section 2, we investigate the structural properties of corona graphs. In Section 3, we focus on spectra, Laplacian spectra and signless Laplacian spectra of corona graphs. Finally, we conclude in Section 4.

*Notation.* In what follows we denote $K_n, P_n, S_n$ and $C_n$ to be the clique, path graph, star graph and circuit on $n$ nodes respectively. By $1_n$ we denote the all-one vector of dimension $n$. The Kronecker product of two matrices is denoted by $\otimes$. Throughout the paper, it is assumed that the corona graphs $G^{(m)}$, $m \geq 1$ are generated by a seed graph $G^{(0)} = G$ which is a simple connected graph on $n$ nodes. The node and edge sets of $G^{(m)}$, $m \geq 0$ are denoted by $V(G^{(m)})$ and $E(G^{(m)})$ respectively, unless stated otherwise.

## 2. Structural properties of corona graphs

In this section, we focus on structural properties of corona graphs. First, we use the definition of corona product of two graphs to define corona graphs as follows.

**Definition 1.** Let $G = G^{(0)}$ be a simple connected graph. Then the corona graphs $G^{(m)}$ corresponding to the seed graph $G$ are defined by

$$G^{(m)} = G^{(m-1)} \circ G, \tag{1}$$

where $m(\geq 1)$ is a natural number.

For example, the corona graphs $G^{(1)}$ and $G^{(2)}$ corresponding to the seed graph $K_3$ are shown in Fig. 1(b) and (c). The following are some immediate obvious observations associated with corona graphs.