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# Graphs and digraphs represented by intervals and circular arcs

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ABSTRACT

In this paper, we have shown that a graph is a circular arc graph if and only if the corresponding symmetric digraph with loops is a circular arc digraph. We characterize circular arc digraphs and circular arc graphs using circular ordering of their edges. We also characterize a circular arc graph as a union of an interval graph and a threshold graph. Finally, we characterize proper interval bigraphs and proper circular arc bigraphs using two linear orderings of their vertex set.



## 1. Introduction

The intersection graph of a family of sets $\mathscr{F} = \{S_v : v \in V\}$ is the graph $G = (V, E)$ with vertices of the graph corresponding to the sets and two vertices are adjacent if and only if the corresponding sets intersect. The families of sets most often considered in connection with intersection graphs are the families of intervals on the real line or arcs of a circle. The corresponding graphs are respectively known as *interval graphs* and *circular arc graphs*. An interval graph (or a circular arc graph) is a *proper interval graph* (or a *proper circular arc graph*) if there is an interval (or a circular arc) representation in which no interval (or circular arc) is properly contained in the other.

Circular arc graphs and its subclasses like *proper circular arc graphs* and *unit circular arc graphs* where all the arcs are of unit length have been extensively studied by Tucker, Hell and others [8,10,11,25–28]. In a survey paper M.C. Lin and J.L. Szwarcfiter [14] summarized several characterizations, recognition algorithms and open problems for these classes of graphs.

Beineke and Zamfirescu [2] and Sen et al. [19] introduced the analogous concept of intersection graph in different contexts. Let $\mathscr{F} = \{(S_v, T_v)\}$ be a collection of ordered pair of sets indexed by a set $V$; we call $S_v$ the *source set* and $T_v$ the *terminal set* for $v \in V$. The intersection digraph of this collection is the digraph with vertex set $V$ with an edge from $u$ to $v$ if and only if $S_u \bigcap T_v \neq \emptyset$. A digraph is an *interval digraph* or a *circular arc digraph* if $\mathscr{F}$ is the family of ordered pair of intervals or arcs of a circle, respectively.

A bipartite graph (in short, bigraph) $B = (X, Y, E)$ is an intersection bigraph if there exists a family $\mathscr{F} = \{S_v; v \in X \cup Y\}$ of sets such that $uv \in E$, if and only if $S_u \bigcap S_v \neq \emptyset$ where $u$ and $v$ belong to opposite partite sets. An intersection bigraph is an *interval bigraph* (or a *circular arc bigraph*) if $\mathscr{F}$ is a family of intervals (or arcs of a circle). The *biadjacency matrix* $A(B)$ of a bigraph $B$ is the submatrix of the adjacency matrix consisting of the rows indexed by one partite set and columns by the other. It has been shown in [5,6] that these two concepts of intersection bigraphs and intersection digraphs are equivalent.

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To characterize circular arc graphs Tucker [25,26] introduced a concept called the "quasi circular ones property" for a 0, 1-matrix. Given a 0, 1-matrix $A$, let $V_i$ be the 1's appearing consecutively in row $i$, starting with the first 1 on or after the main diagonal and wrapping around (if possible) until the first 0 is reached and $W_j$ be the 1's appearing consecutively in column $j$, starting with the first 1 on or after the main diagonal and wrapping around (if possible) until the first 0 is reached. Then $A$ has the *quasi circular ones property* if the union of the sets $V_i$ and the sets $W_j$ cover all the 1's in $A$.

In particular, the augmented adjacency matrix $A^*(G)$ of a graph $G$ be the adjacency matrix $A(G)$ of $G$ together with 1's in the main diagonal. For $A^*(G)$ the sets $V_i$ and $W_j$ all start at the main diagonal. Tucker [25,26] proved that $G$ is a circular arc graph if and only if its vertices can be indexed so that $A^*(G)$ has quasi circular ones property.

A *stair partition* of a matrix is a partition of its position into two sets $(L, U)$ by a polygonal path from the upper left to the lower right, such that $L$ is closed under leftward or downward movement, and the set $U$ is closed under rightward or upward movement. To give an adjacency matrix characterization of circular arc digraphs, Sen et al. [20] defined generalized circular one property of a binary matrix as follows: Given a 0, 1-matrix $A$ and a stair partition $(L, U)$, let $V_i$ be the $1's$ in row $i$ that begins at the stair and continue rightward (around if possible) until first 0 is reached. Also let $W_j$ be the $1's$ in column $j$ that begins at the stair and continue downward (around if possible) until first 0 is reached. Then $A$ has the *generalized circular one property* if it has a stair partition $(L, U)$ such that the $V_i's$ and $W_j's$ together cover all $1's$ of $A$. They proved that $D$ is a circular arc digraph if and only if $A(D)$ has independent row and column permutation such that $A(D)$ has the generalized circular one property. In this paper we give a shorter proof of this result.

To characterize interval graph we introduce the notion of quasi linear ones property for a 0, 1-matrix. Given a 0, 1-matrix $A$, let $V_i$ be the 1's appearing consecutively in row $i$, starting with the first 1 on or after the main diagonal until the first 0 is reached. Also let $W_j$ be the 1's appearing consecutively in column $j$, starting with the first 1 on or after the main diagonal until the first 0 is reached. Then $A$ has the *quasi linear ones property* if the union of the set $V_i$ and the set $W_j$ cover all the 1's in $A$. It may be noted that an augmented adjacency matrix satisfying quasi linear ones property is same as the quasi diagonal matrix of Mirkin and Rodin [17].

In [22] it was shown that an interval digraph is actually a generalization of an interval graph in the sense that an undirected graph $G$ is an interval graph if and only if the corresponding symmetric digraph $D(G)$ with loops at all the vertices is an interval digraph. We have shown that this result is also true for circular arc graphs.

In [18] an interval digraph was characterized in terms of vertex–edge incidence matrix suitably defining a linear ordering of the edges of the digraph. Analogously to characterize a circular arc digraph in terms of vertex–edge incidence matrix, we define an ordering $(<)$ of the edges of the digraph, regarded the incidence matrix as wrapped around a right circular cylinder, such that for $uv, rs, ut, pv \in E$

(i) $uv < rs < ut \Longrightarrow us \in E (v \neq t)$ also

(ii) $uv < rs < pv \Longrightarrow rv \in E (u \neq p)$.

($uv \in E \Longrightarrow$ the directed edge from $u$ to $v$).

We call such an ordering of the edges of the incidence matrix a *consistent circular ordering*. We shall prove that a digraph $D = (V, E)$ is a circular arc digraph if and only if the edge set $E$ has a consistent circular ordering.

It can be observed that making appropriate changes in the definition of consistent circular ordering and primitives (for definition see the proof of Theorem 4), we obtain an analogous characterization for circular arc graphs.

It can be observed that if we cut the circle at an arbitrary point, then a circular arc bigraph becomes an interval bigraph. Dwelling on this idea Basu et al. [1] showed that a circular arc bigraph can be characterized as a union of an interval bigraph and a related Ferrers bigraph.

To give analogous characterization for circular arc graphs, first we recall the definition of a threshold graph.

A graph $G = (V, E)$ is called a *threshold graph* when there exist non negative reals $w_v$, $v \in V$ and $t$ such that $\sum_{v \in U} w_v \leq t$ if and only if $U$ is a stable set and $U \subseteq V$.

There are several characterizations of threshold graph [9,15]. Below we give two of them.

**Theorem 1** *([9,15]). For a graph $G = (V, E)$, the following conditions are equivalent:*

*(a) $G$ is a threshold graph;*

*(b) $G$ does not contain $P_4$, $C_4$, $2K_2$ as an induced subgraph;*

*(c) $G$ is a split graph $G(K, S)$ (i.e. its vertices can be partitioned into a clique $K$ and a stable set $S$) and the neighbors of the vertices of $S$ are nested.*

In Theorem 8 we shall characterize a circular arc graph as a union of an interval graph and a threshold graph. As a consequence, a proper circular arc graph can be characterized as a union of a proper interval graph and a threshold graph.

As mentioned earlier the class of intersection digraph and intersection bigraph are equivalent concept. The class of interval bigraphs and circular arc bigraphs have been extensively studied by several researchers [6,11–13,18,20,22]. A binary matrix, (i.e. a 0, 1-matrix) has the *partitionable zeros property*, if each 0 can be replaced by one of $\{R, C\}$ in such a way that every $R$ has only $R's$ to its right and every $C$ has only $C's$ below it. In [20], an interval bigraph $B$ was characterized in terms of partitionable zeros property of its biadjacency matrix $A(B)$.

An interval bigraph $B = (X, Y, E)$ is a *proper interval bigraph* if no interval corresponding to $X$-partite (or, $Y$-partite) set properly contains another interval. Recently Lundgren and Brown [3] generalized this definition as follows.

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