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Axiomatic characterization of the median and antimedian function on a complete graph minus a matching

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ABSTRACT

A median (antimedian) of a profile of vertices on a graph G is a vertex that minimizes (maximizes) the sum of the distances to the elements in the profile. The median (antimedian) function has as output the set of medians (antimedians) of a profile. It is one of the basic models for the location of a desirable (obnoxious) facility in a network. The median function is well studied. For instance it has been characterized axiomatically by three simple axioms on median graphs. The median function behaves nicely on many classes of graphs. In contrast the antimedian function does not have a nice behavior on most classes. So a nice axiomatic characterization may not be expected. In this paper an axiomatic characterization is obtained for the median and antimedian function on complete graphs minus a matching.

1. Introduction

Facility location problems in discrete location theory deal with functions that find an appropriate location for a common facility or resource in a graph. The main objective is to minimize the cost of accessing a facility or sharing a resource in the graph. Typical problems of this kind that have been studied extensively are: (i) The *median problem*: finding a vertex that minimizes the distance sum to the clients. (ii) The *mean problem*: finding a vertex that minimizes the sum of the squares of the distances to the clients. (iii) The *center problem*: minimizing the maximum distance to the clients. The first two problems can be used to model finding the optimal location for a distribution center. The last problem can be used to model finding the optimal location for a fire station. The *antimedian problem* is a different type of location problem in which the facility is of obnoxious nature (i.e. the clients want to have it as far away as possible), for example a garbage dump. In this case minimizing the 'cost' is equivalent to maximizing the distance sum.

A consensus function is used to model the problem of achieving consensus amongst agents or clients in a rational way. The input of the consensus function is information on the clients and the output concerns the issue on which consensus should be

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reached. To guarantee the rationality of the process, the consensus function satisfies certain rules called 'consensus axioms'. Such axioms should be appealing and simple. Of course, this depends on the consensus function at hand. A function with nice properties might be characterized by simple axioms. But a function that behaves badly might need more complicated or less appealing axioms. K. Arrow initiated the study of the axiomatics of consensus functions in his seminal paper [1] of 1951. For more references in this area see [2,3,17].

Location problems can also be viewed as consensus problems. Then one wants to characterize the location function by a set of axioms. Holzman [10] was the first to study location functions from this perspective. His focus was on the mean function on a tree network (the continuous variant of a tree, where internal points of edges are also allowed as location). Then Vohra [27] characterized the median function axiomatically on tree networks. The discrete case was first dealt with by McMorris, Mulder and Roberts [16]: the median function on cube-free median graphs was characterized using three simple and appealing axioms, see below. The mean function and the ℓ_p-function on trees (discrete case) were first characterized by McMorris, Mulder and Ortega in [14] and [15], respectively. The center function on trees has been characterized by McMorris, Roberts and Wang [18], see also [24]. Recently the median function has been characterized on hypercubes and median graphs by Mulder and Novick [22,23] using the same three simple axioms as in [16].

One can expect that the axioms may depend on the consensus function at hand and on the structure of the graph on which the function is studied. The case of the median function on median graphs turns out to be exceptional: in this case the three axioms in [16,22,23] hold for the median function on any finite metric space, but on median graphs they actually characterize the median function. This is due to the very rich structure of median graphs, see e.g. [11,20,21].

So far, in all other instances mentioned above, a characterization is only obtained on trees. Moreover, besides some axioms that hold on any connected graph for the respective function, always some other axioms are involved that actually depend on the graph being a tree.

We focus on the characterization of two location functions: the median function and the antimedian function. The antimedian function maximizes the sum of the distances to the clients, see e.g. [19,4–7,25]. The differences between these two functions are quite striking. A first inspection of the antimedian function already shows that, even on trees, it does not behave nicely at all, let alone on arbitrary graphs. Only on special classes, such as paths, hypercubes and complete graphs, does it seem to have a nice behavior. The axiomatization of the antimedian function on hypercubes and paths is well studied in [8]. The median and antimedian function on cocktail-party graphs and antimedian function on complete graphs are characterized in [9]. In this paper we generalize this characterization to complete graphs minus a matching. Cocktail-party graphs and complete graphs are special cases in this graph class. A cocktail-party graph is a complete graph of even order minus a perfect matching.

In Section 2 we set the stage. In Section 3 we characterize the median function on complete graphs minus a matching by a set of four axioms. In Section 4 we characterize the antimedian function on the same graph by another set of five axioms. For the case of complete graphs only four axioms are needed. We also study the independence of the axioms in these characterizations. For axiomatic characterizations of the median function on complete graphs we refer to [26,12].

2. Preliminaries

Let $G = (V, E)$ be a finite, connected, simple graph with vertex set V and edge set E. The distance $d(u, v)$ between u and v in G is the length of a shortest u, v-path. The interval $I(u, v)$ between two vertices u and v in G consists of all vertices on shortest u, v-paths, that is:

$$I(u, v) = \{x \mid d(u, x) + d(x, v) = d(u, v)\}.$$

A profile π of length $k = |\pi|$ on G is a nonempty sequence $\pi = (x_1, x_2, \ldots, x_k)$ of vertices of V with repetitions allowed. We define V^* to be the set of all profiles of finite length on V. We call $x_1, x_2, \ldots, x_k$ the *elements* of the profile. A *vertex* of π is a vertex that occurs as an element in π. By $\{\pi\}$ we denote the set of all vertices of π. Note that a vertex may occur more than once as element in π. If we say that x is an *element* of π, then we mean an element in a certain position, say $x = x_j$ in the jth position. A *subprofile* of π is just a subsequence of π. For convenience we allow a subprofile to be empty. Let $\pi = (x_1, x_2, \ldots, x_k)$ and $\rho = (y_1, y_2, \ldots, y_\ell)$ be two profiles. The profile $(x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_\ell)$ is called a *concatenation* of π and ρ, and is denoted by $\pi\rho$. Note that in most cases we have $\pi\rho \neq \rho\pi$. The profile consisting of the concatenation of m copies of π is denoted by π^m. Let π be a profile on G. A vertex with highest occurrence in π is called a *plurality vertex* of π. We denote the set of plurality vertices of π by $Pl(\pi)$.

A *consensus function* on G is a function $F : V^* \to 2^V - \emptyset$ that gives a nonempty subset of V as output for each profile on G. For convenience, we write $F(x_1, \ldots, x_k)$ instead of $F((x_1, \ldots, x_k))$, for any function F defined on profiles, but will keep the brackets where needed.

The *remoteness* of a vertex v to a profile π is defined as

$$r(v, \pi) = \sum_{i=1}^{k} d(x_i, v).$$

A vertex minimizing $r(v, \pi)$ is called a *median* of the profile. The set of all medians of π is the *median set* of π and is denoted by $M(\pi)$. A vertex maximizing $r(v, \pi)$ is called an *antimedian* of the profile. The set of all antimedians of π is the