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$\vec{P}_3$-decomposition of directed graphs

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ABSTRACT

A $\vec{P}_3$-decomposition of a directed graph D is a partition of the arcs of D into directed paths of length 2. We characterize symmetric digraphs that do not admit a $\vec{P}_3$-decomposition. We show that the only 2-regular, connected directed graphs that do not admit a $\vec{P}_3$-decomposition are obtained from undirected odd cycles by replacing each edge by two oppositely directed arcs. In both cases, we give a linear-time algorithm to find a $\vec{P}_3$-decomposition, if it exists.

1. Introduction

Let G be a graph and $G_1, G_2, \ldots, G_k$ be subgraphs of G. We say that the collection of subgraphs $G_1, G_2, \ldots, G_k$ is a decomposition of the graph G, if every edge in G is an edge in exactly one of the subgraphs. In other words, the subgraphs G_i are pairwise edge-disjoint, and their union is the graph G. If $\mathcal{F} = \{F_1, F_2, \ldots, F_r\}$ is a family of graphs, an $\mathcal{F}$-decomposition of G is a decomposition of G into subgraphs, each of which is isomorphic to some graph in $\mathcal{F}$. If $\mathcal{F}$ contains a single graph H, an $\mathcal{F}$-decomposition is called an H-decomposition. In particular, a P_3-decomposition of a graph G is a partition of the edge set of G into paths of length 2.

The same notion of decomposition applies to directed graphs D as well, where we require each arc in D to be contained in exactly one of the subgraphs in the decomposition. A $\vec{P}_3$-decomposition of a directed graph D is a partition of the arc set of D into directed paths of length 2.

A classical result on graph decomposition, originally noted by Kotzig [3], but now considered a simple exercise, is the following.

Theorem 1. *A graph G has a P_3-decomposition iff every connected component of G has an even number of edges.*

However, no such characterization of directed graphs that admit a $\vec{P}_3$-decomposition is known, and the problem does not appear to have been studied much. This question is more difficult as even a simple directed graph may contain cycles of length 2, and two adjacent arcs do not necessarily form a $\vec{P}_3$. The problem of characterizing multigraphs that admit a P_3-decomposition was raised in [5], and some partial results were obtained in [1,2,4]. However, even this does not appear to have been solved in general, and we do not know of any such work for directed graphs.

It may be noted that all these problems can be solved in polynomial-time, by a reduction to the perfect matching problem. Given a graph (multigraph, directed graph), we construct a new graph whose vertices are the edges (arcs) in the given graph, and two vertices are adjacent in the new graph iff the corresponding edges (arcs) induce a path (directed path) of length 2

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in the original graph. A $P_3(\vec{P}_3)$-decomposition of the given graph (multigraph, directed graph) then corresponds to a perfect matching in the new graph, and vice versa.

While Tutte's 1-factor theorem [6] gives a necessary and sufficient condition for a graph to have a perfect matching, we would like a simple condition on the original graph (multigraph, directed graph) that guarantees a $P_3(\vec{P}_3)$-decomposition. Theorem 1 gives such a condition for graphs. Further, while a perfect matching can be found in polynomial-time, we would like a simpler algorithm to find a $P_3(\vec{P}_3)$-decomposition. The proof of Theorem 1 also gives a simple linear-time algorithm to find a P_3-decomposition of graphs.

A directed graph is said to be *symmetric* if for every pair of distinct vertices u, v, there is an arc from u to v iff there is an arc from v to u. In other words, the directed graph is obtained by replacing each edge in an undirected graph by two oppositely directed arcs. Our main result is a characterization of symmetric directed graphs that do not admit a $\vec{P}_3$-decomposition. The characterization is an explicit constructive characterization that describes the structure of symmetric directed graphs that do not admit a $\vec{P}_3$-decomposition. This also leads to a simple linear-time algorithm to find a $\vec{P}_3$-decomposition of a symmetric directed graph, if it exists.

A directed graph is 2-regular if every vertex has indegree and outdegree 2. We show that the only 2-regular, connected directed graphs, not necessarily symmetric, that do not admit a $\vec{P}_3$-decomposition, are obtained from undirected odd cycles by replacing each edge by two oppositely directed arcs. Again, we give a linear-time algorithm to find a $\vec{P}_3$-decomposition of all other 2-regular directed graphs.

In this paper, for ease of using induction, we will consider graphs that may have loops as well as multiple edges, and we will call them pseudo-graphs to emphasize the fact. A simple graph will be referred to as a graph. We assume that each edge in an undirected pseudo-graph has two ends, which could be the same vertex, if the edge is a loop. An edge with ends u, v will be denoted $\{u, v\}$. Similarly, every arc in a directed pseudo-graph will have two ends, one of which is called the head, and the other the tail. An arc with tail u and head v will be denoted (u, v).

We will consider decompositions of directed pseudo-graphs into subgraphs with two arcs that form a trail. Any subgraph in such a decomposition will be denoted simply by the sequence of two arcs it contains. The family $\mathcal{F}$ of these directed pseudo-graphs is defined below.

Definition 1. Let $\mathcal{F}$ be the family of all directed pseudo-graphs with no isolated vertices, and 2 arcs that induce a trail of length 2. Thus $\mathcal{F}$ contains the five directed pseudo-graphs with arcs (u, u), (u, u) and (u, u), (u, v) and (u, v), (v, v) and (u, v), (v, u) and (u, v), (v, w).

All other definitions and notations are standard.

2. Symmetric directed graphs

If G is a pseudo-graph, let $D(G)$ denote the directed pseudo-graph obtained from G by replacing each edge $e = \{u, v\}$ in G, by two oppositely directed arcs, (u, v) and (v, u). We will call one of these arcs e^+ and the other e^- arbitrarily. Note that a loop in G is replaced by two loops in $D(G)$.

Let G be any pseudo-graph. An $\mathcal{F}$-decomposition of $D(G)$ is said to be *compatible*, if for every edge e in G, the arcs e^+ and e^- are contained in different subgraphs in the decomposition of $D(G)$.

Definition 2. Let $\mathcal{G}$ be the minimal set of pseudo-graphs that satisfies the following properties.

1. The two connected pseudo-graphs with one edge, $\{u, u\}$ and $\{u, v\}$, are in $\mathcal{G}$.
2. If G is a pseudo-graph in $\mathcal{G}$ and e an edge in G, then the pseudo-graph obtained by subdividing twice the edge e is in $\mathcal{G}$. In other words, if $e = \{u, v\}$, then the pseudo-graph obtained from G by deleting the edge e, adding two new vertices x, y, and the edges $\{u, x\}$, $\{x, y\}$, $\{y, v\}$, is in $\mathcal{G}$.
3. If G is a pseudo-graph in $\mathcal{G}$ and v a vertex in G, then the pseudo-graph obtained from G by adding a new vertex x, and edges $\{v, x\}$, $\{x, x\}$ is in $\mathcal{G}$.
4. If G is a pseudo-graph in $\mathcal{G}$ and v a vertex in G, then the pseudo-graph obtained from G by adding two new vertices x, y, and edges $\{v, x\}$, $\{x, y\}$ is in $\mathcal{G}$.

We are now ready to state the main result.

Theorem 2. *For any connected pseudo-graph G, $D(G)$ has a compatible $\mathcal{F}$-decomposition iff $G \notin \mathcal{G}$.*

We note the following corollary.

Corollary 1. *For any connected graph G, $D(G)$ has a $\vec{P}_3$-decomposition iff $G \notin \mathcal{G}$.*

Proof. This follows from Theorem 2, since if G is a graph, then $D(G)$ does not have any loops, and any 2-cycle in $D(G)$ contains arcs e^+, e^- for some edge e in G. Therefore, any compatible $\mathcal{F}$-decomposition of $D(G)$ is in fact a $\vec{P}_3$-decomposition, and vice versa. □