ARTICLE IN PRESS

Discrete Applied Mathematics ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Solving Hamiltonian Cycle by an EPT algorithm for a non-sparse parameter[☆]

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ARTICLE INFO

Article history:
Received 30 June 2015
Received in revised form 5 February 2016
Accepted 10 February 2016
Available online xxxx

Keywords:
Hamiltonian Cycle
Parameterized complexity
Split-matching-width
EPT

ABSTRACT

Many hard graph problems, such as Hamiltonian Cycle, become FPT when parameterized by treewidth, a parameter that is bounded only on sparse graphs. When parameterized by the more general parameter clique-width, Hamiltonian Cycle becomes W[1]-hard, as shown by Fomin et al. (2010). Sæther and Telle address this problem in their paper (Sæther and Telle, 2016) by introducing a new parameter, split-matching-width, which lies between treewidth and clique-width in terms of generality. They show that even though graphs of restricted split-matching-width might be dense, solving problems such as Hamiltonian Cycle can be done in FPT time.

Recently, it was shown that Hamiltonian Cycle parameterized by treewidth is in EPT (Bodlaender et al., 2013; Fomin et al., 2014), meaning it can be solved in $n^{\mathcal{O}(1)}2^{\mathcal{O}(k)}$-time. In this paper, using tools from Fomin et al. (2014), we show that also parameterized by split-matching-width Hamiltonian Cycle is EPT. To the best of our knowledge, this is the first EPT algorithm for any "globally constrained" graph problem parameterized by a non-trivial and non-sparse structural parameter. To accomplish this, we also give an algorithm constructing a branch decomposition approximating the minimum split-matching-width to within a constant factor. Combined, these results show that the algorithms in Sæther and Telle (2016) for Edge Dominating Set, Chromatic Number and Max Cut all can be improved. In fact, using our new approximation algorithm, under the Exponential Time Hypothesis, the Hamiltonian Cycle algorithm of this paper, and the three algorithms for MaxCut, Edge Dominating Set, and Chromatic Number, are asymptotically optimal.

1. Introduction

The problem of finding a Hamiltonian Cycle in a graph – a simple cycle covering all the vertices of the graph – is NP-complete [10]. One way to handle an NP-hard problem is by investigating its parameterized complexity, for various choices of parameter. Unlike a lot of other NP-hard graph problems, Hamiltonian Cycle does not have a natural parameter, since the solution size is the number of vertices in the input graph. Instead, we may look at structural parameterizations of the input graph, for instance its treewidth or clique-width.

A lot of NP-hard graph problems become fixed parameter tractable (FPT, solvable in $f(k)n^{\mathcal{O}(1)}$-time for parameter-value k) when parameterized by treewidth. Many examples of problems that can be checked locally, e.g., Independent Set, Vertex Cover, Dominating Set and so on, are even EPT when parameterized by treewidth, meaning that the problems can be solved in time $2^{\mathcal{O}(k)}n^{\mathcal{O}(1)}$ [4] (also referred to as having a single exponential algorithm). When parameterized by clique-width,

☆ This work is supported by the Research Council of Norway, project PARALGO.

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http://dx.doi.org/10.1016/j.dam.2016.02.008
0166-218X/

Please cite this article in press as: S.H. Sæther, Solving Hamiltonian Cycle by an EPT algorithm for a non-sparse parameter, Discrete Applied Mathematics (2016), http://dx.doi.org/10.1016/j.dam.2016.02.008

hardly any of these problems are known to be EPT. For instance, Dominating Set has recently been shown solvable in time $2^{\mathcal{O}(k \log k)} n^{\mathcal{O}(1)}$ for clique-width k [12], but this is still not EPT.

For problems that have a global constraint, like Steiner Tree, Hamiltonian Cycle and Feedback Vertex Set, EPT algorithms parameterized by treewidth were for a long time not known. For example, the asymptotically best algorithm for Hamiltonian Cycle was for a long time the folklore $n^{\mathcal{O}(1)} k^{\mathcal{O}(k)}$ time algorithm, resulting in a belief that graph problems with a global requirement may not have EPT algorithms. Recently, however, a breakthrough paper by Cygan et al. [3] gave a randomized EPT algorithm for Hamiltonian Cycle, and other problems with global constraints, when parameterized by treewidth. Shortly after this, Bodlaender et al. [1] and then Fomin et al. [6], also found deterministic EPT algorithms for Hamiltonian Cycle parameterized by treewidth. Both the papers [1,6] are general, in the sense that they provide a framework for solving many problems. Graph classes of bounded treewidth are all sparse, so one may wonder if using either of these new frameworks will help in finding similar EPT results for globally constrained problems like Hamiltonian Cycle for a parameter bounded also on non-sparse graph classes. The classical structural graph parameter bounded also on some non-sparse graphs is clique-width. Unfortunately, it is unlikely that such a result exists for clique-width, as Hamiltonian Cycle has been shown to be W-hard when parameterized by clique-width [5]. So, we must focus on a non-sparse parameter which is less general than clique-width. Examples of some such parameters are modular-width, shrub-depth, neighbourhood diversity, twin-cover, and the newly introduced split-matching-width (see Fig. 1).

In the recent paper [7] Gajarský et al. give an FPT algorithm (but not EPT) for Hamiltonian Cycle parameterized by modular-width, and show W-hardness when parameterized by shrub-depth. Split-matching-width is a new parameter introduced by Sæther and Telle [13] for which Hamiltonian Cycle is FPT [13]. Unlike modular-width, split-matching-width generalizes treewidth, so it is a good candidate for applying the framework used for treewidth (see Section 2 for the definition of split-matching-width).

In this paper, we will show that using the framework of [6] we can solve Hamiltonian Cycle in time $2^{\mathcal{O}(k)} n^{\mathcal{O}(1)}$ for parameter k being split-matching-width. The approach will be similar to that of [13] in the sense that it consists of two parts; (1) given a graph G, finding a branch decomposition of low split-matching-width, and then (2) solving Hamiltonian Cycle on G with a runtime depending on the split-matching-width of the computed branch decomposition. We will in this paper improve on the results from [13] by showing the following two theorems that when combined results in an EPT algorithm for Hamiltonian Cycle parameterized by split-matching-width.

Theorem 1. *Given a graph G of split-matching-width less than k, in $n^{\mathcal{O}(1)} 2^{\mathcal{O}(k)}$ time we can find a branch decomposition of split-matching-width less than* $16k$.

Theorem 2. *Given a graph G and a branch decomposition of split-matching-width k, we can decide if G has a Hamiltonian Cycle in time $n^{\mathcal{O}(1)} 2^{\mathcal{O}(k)}$.*

Another result of Theorem 1 is that we can improve the runtime of the algorithms for solving Edge Dominating Set, Chromatic Number, and Max Cut parameterized by split-matching-width described in [13]. In fact, under the Exponential Time Hypothesis the asymptotic runtimes for Max Cut, Hamiltonian Cycle, Chromatic Number and Edge Dominating Set become optimal [8,9]. (I.e., no $n^{\mathcal{O}(1)} 2^{o(k)}$ algorithm exists.)

This paper is organized as follows: In Section 2, we give the necessary definitions and background needed for the rest of the paper. In Section 3 we prove Theorem 2. We start the section by stating one of the powerful results found in [6], then use the rest of the section to adapt this result to be used in the same type of dynamic programming scheme as found in [13]. In Section 4 we show how to prove Theorem 1. We give an idea of how a graph can be decomposed into so called prime graphs, and how a branch decomposition for the original graph can be made out of combining decompositions of these prime graphs. The rest of the section then shows how a modified version of mm-width and sm-width, which we call lifted mm-width/sm-width, will help us improve the approximation of [13] by better tying the width of these prime graphs to the width of the original graph than what was done in [13]. The paper ends with Section 5 where we give a short summary.

2. Preliminaries and terminology

Graph and set preliminaries

We work on simple undirected graphs $G = (V, E)$ and denote the set of vertices and set of edges of a graph G by $V(G)$ and $E(G)$, respectively. We use n to denote the number of vertices of the graph in question. For an edge between vertices u and v, we simply write uv. For a path P, by writing uPv we mean that the endpoints of P are u and v. For a graph G and subset $A \subseteq V(G)$, we denote by $G[A]$ the subgraph of G *induced* by A. That is, the vertex set $V(G[A])$ of $G[A]$ is A and the edge set is $E(G[A]) = \{uv \in E(G) : u, v \in A\}$. For disjoint sets $A, B \subseteq V(G)$, we denote by $G[A, B]$ the bipartite subgraph of G induced by the pair (A, B). That is $V(G[A, B]) = A \cup B$ and $E(G[A, B]) = \{uv \in E(G) : u \in A, v \in B\}$. For a set of vertices $S \subseteq V(G)$, we denote by $N_G(S)$ all the vertices in $V(G) \setminus S$ adjacent to S. We omit the subscript G in $N_G(S)$ when it is clear from context. For a single vertex v, we write $N_G(v)$ instead of $N_G(\{v\})$. To *contract* an edge uv means to replace the vertices u and v by a new vertex v_{uv} adjacent to exactly the same vertices as u and v combined. For a set $A \subseteq V(G)$, when $V(G)$ is clear from context, we write $\overline{A}$ to mean the set $V(G) \setminus A$. For a graph G and subsets $A, B, C \subseteq V(G)$, we say that C separates A and B if there are no paths from $A \setminus C$ to $B \setminus C$ in $G[\overline{C}]$.

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