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Multi-player Wythoff's game and its variants

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ABSTRACT

Wythoff's game is a well-known 2-player impartial combinatorial game, introduced by W.A. Wythoff in 1907. In recent years, many scholars studied the variants of Wythoff's game, including mainly extensions and restrictions, with fruitful results achieved. One way of solving n-player impartial games was presented by W.O. Krawec in 2012. We employ Krawec's function in this paper to analyze n-player Wythoff's game and its nine restricted versions. The game values are completely determined for all ten n-player impartial games.

1. Introduction

We assume that the reader has some knowledge in combinatorial game theory. Basic definitions can be found in [2,5]. *Wythoff's game* [29] is a well-known 2-person perfect information game. The game is played with two heaps of tokens. Two players move alternately. Each player can either remove any number of tokens from a single heap (*Nim rule*) or remove the same number of tokens from both heaps (*Wythoff's rule*). All P-positions of Wythoff's game under normal play convention were given in [29]. All P-positions of Wythoff's game under the misère play convention were determined in [10].

In many papers devoted to variations of Wythoff's game, new rules are adjoined to the original ones. Such variations are called *extensions*. For the games of this type, we refer the reader to *a-Wythoff's game* [9], (s, t)*-Wythoff's game* [11,21,25], *Wythoff-like game* [14,24], *Geometrical extensions* [8], *two-parameter extension* [1], etc.

There are a few papers where only subsets of Wythoff's moves are allowed. Such variations are called *restrictions*. See [4,6,7,13,22]. For all these extensions and restrictions of Wythoff's game, the main goal is to find characterizations of the sequence of P-positions, which almost always differs from the original Wythoff's sequence [7,22].

Naturally it is of interest to generalize as much as possible of the theory of 2-player games to n-player games. Different from 2-player perfect information games, when there are more than two players, it may so happen that one of the players can help any of the players to win, but anyhow, he himself has to lose. So the outcome of the game depends on how the group coalitions are formed among the players. In previous literature, several possibilities were investigated:

- *Multi-player without alliance*. See [3,19,26–28].
- *Multi-player with two alliances*. See [15,16,30].
- *Multi-player with alliance system*. W.O. Krawec ([17], 2012) assumed that every player has a fixed set of allegiances to all n players i.e. an *alliance system* may be defined arbitrarily before the start of a game. While the alliance system used is fixed for the duration of the game, Krawec provided a method of analyzing n-player impartial games, and derived a recursive function capable of determining which of the n players has a winning strategy.

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W.O. Krawec ([18], 2015) developed a method of analyzing n-player impartial combinatorial games where $n-1$ players behave optimally while one of the players plays randomly i.e. one player chooses games at random without strategy.

W.A. Liu and M.Y. Wang ([23], 2017) analyzed "Multi-player subtraction games". Krawec's result was generalized from $n=3$ to an arbitrary integer $n \geq 3$, and the order of subtraction games from $k=2$ to an arbitrary integer $k \geq 2$. The 3-player subtraction games of order 2 were completely analyzed. It turns out that the sequences of game values are always periodic. The explicit representations of pre-periods and periods were presented.

W.A. Liu and J.W. Duan ([20], 2017) analyzed "Misère N-pile Nim with n players", abbreviated by MiNim(N, n). The game values of MiNim(N, n) are completely determined for three cases $n > N+1$, $n = N+1$ and $n = N$.

Definition 1. We introduce a class of *n-player Wythoff's game*, denoted by $W(n)$: By (x, y) we denote a position of the two piles of size x and y. There are n players $P_0, P_1, \ldots,$ and P_{n-1}. Assume that P_0 is the first player to move, then P_1 and so on. After player P_{n-1}'s turn, P_0 will play again. For any given position (x, y), a legal move is one of the following two types:

(Nim move) Removing a positive number of tokens from a single pile (possibly an entire pile) i.e.

$$(x, y) \longrightarrow (x-a, y) \text{ with } 1 \leq a \leq x, \text{ or } (x, y) \longrightarrow (x, y-b) \text{ with } 1 \leq b \leq y. \quad (1)$$

(Diagonal move) Removing an equal positive number of tokens from both piles i.e. $(x, y) \longrightarrow (x-c, y-c)$ and

$$1 \leq c \leq \min\{x, y\}. \quad (2)$$

The first player who cannot make any move wins.

Symmetry of the game rules implies that (x, y) and (y, x) have the same winner i.e. a position can be represented by (x, y) with $y \geq x$. Throughout this paper, we adopt a particular alliance system as defined by Krawec's model. We will give our attention to the following questions:

Question 1. Can we determine the winner of $W(n)$ for all integers $n \geq 3$ and all positions (x, y) with $y \geq x \geq 1$?

In Section 3, Theorem 7 gives a positive answer. It turns out that, regardless of the number of players n, the winner will be P_0, P_1 or P_2, and that only $n = 3$ or $n > 3$ matters:

(i) If $n > 3$, P_1 or P_2 will be the winner and the player P_i, $i \in \{0, 3, 4, \ldots, n-1\}$, has no chance to win. More clearly, the winner is P_1 if $y = x \geq 1$, P_2 if $y > x \geq 1$.

(ii) If $n = 3$, each of P_0, P_1 and P_2 has chance to win which depends on the relation between the parameters x and y. More clearly, if $x = 1$ and $y = 2$, the winner is P_2; if $(x = 1, y \geq 3)$ or $2 = x \leq y$ or $(x \geq 3, y = x+1)$, the winner is P_0; if $(x = 1, y = 1)$ or $y = x \geq 3$ or $(x \geq 3, y > x+1)$, the winner is P_1.

Question 2. Does there exist a restricted version Γ of $W(n)$ such that the winner of Γ is the same with that of $W(n)$ for all positions (x, y) with $y \geq x \geq 1$ (i.e. the winner remains unchanged)?

Section 4 is devoted to Question 2. We define nine models where each model is a restricted version of $W(n)$. The winner of each of these nine games is completely determined. Based on these results, we find three classes of restricted versions of $W(n)$ which give Question 2 a positive answer. Moreover, the winner of each of these nine games is still P_0, P_1 or P_2. The following question is natural.

Question 3. Is there a restricted version Γ^* of $W(n)$ (beyond the nine restricted versions considered above) where other players could win (i.e. a player P_i, $i \geq 3$, will be winner of Γ^*)?

The answer is positive! Such a model is given in Section 5(III). We also analyze the method of finding other models of this kind of the expected property.

Question 4. Our results show that for any model of $W(n)$ and the nine restricted versions, the winner can be completely determined by only considering $n = 3$ or $n > 3$. How this phenomenon changes if the probabilistic model cited in [18] is used?

We partly analyze the probabilistic version of $W(n)$ in Section 5(IV). It turns out that we must consider three cases $n = 3$, $n = 4$ and $n > 4$. This means that the probabilistic version of $W(n)$ is different from the original $W(n)$.

2. Basic definitions

Throughout the paper, we employ some definitions and notation used by [17,18].

Definition 2. An *n-player impartial game* is defined to have the following characteristics:

(1) There is n-player who take turns in sequential unchanging order.
(2) There are finitely many positions and these positions are clearly defined to players at all times.
(3) The first player who cannot make any legal move wins.
(4) The game must always end at some point with a clear winner.
(5) There is no difference between the moves allowed to each player.
(6) There are no chance moves.

In this definition, the winner of an n-player game is declared the first player who cannot make a legal move (i.e. the misère play convention), as opposed to the typical two-player version where such a player is declared the loser under the normal play convention.