ARTICLE IN PRESS

Discrete Applied Mathematics ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Complexity and computation of connected zero forcing

Boris Brimkov *, Illya V. Hicks

Computational & Applied Mathematics, Rice University, Houston, TX 77005, USA

ARTICLE INFO

Article history:
Received 7 August 2016
Received in revised form 20 May 2017
Accepted 31 May 2017
Available online xxxx

Keywords:
Connected zero forcing
Zero forcing
NP-complete
Unicyclic graph
Cactus graph
Block graph

ABSTRACT

Zero forcing is an iterative graph coloring process whereby a colored vertex with a single uncolored neighbor forces that neighbor to be colored. It is NP-hard to find a minimum zero forcing set – a smallest set of initially colored vertices which forces the entire graph to be colored. We show that the problem remains NP-hard when the initially colored set induces a connected subgraph. We also give structural results about the connected zero forcing sets of a graph related to the graph's density, separating sets, and certain induced subgraphs. Finally, we give efficient algorithms to find minimum connected zero forcing sets of unicyclic graphs and variants of cactus and block graphs.

1. Introduction

Zero forcing is an iterative graph coloring process where at each time step, a colored vertex with a single uncolored neighbor forces that neighbor to become colored; the zero forcing number of a graph is the cardinality of the smallest set of initially colored vertices which causes the entire graph to be colored. Zero forcing was introduced in a 2006 AIM workshop on linear algebra and graph theory [2] and was used to bound the maximum nullity (equivalently, the minimum rank) of the family of symmetric matrices associated with a graph. The zero forcing number is generally more attainable than the maximum nullity, which makes it a valuable tool in the study of this algebraic parameter. Zero forcing was also independently studied in quantum physics [15] and theoretical computer science [56]; subsequently, applications of zero forcing in logic circuits [16], power network monitoring [35,58], and modeling the spread of diseases and information in social networks [19,24] have also been explored. Since computing the zero forcing number is NP-complete [1,56], the majority of research in this area has focused on developing structural results on zero forcing sets [2,25,36,49], bounding the zero forcing number [22,32,38], relating the zero forcing number to other graph parameters [6,9,53], and characterizing the zero forcing numbers of graphs with special structure [27,44,57].

A natural graph theoretic variant of zero forcing is obtained by requiring every set of initially colored vertices to induce a connected subgraph. This variant of zero forcing, called connected zero forcing, can further the understanding of the zero forcing process and the underlying structure of zero forcing sets in general. Requiring a zero forcing set to be connected also has meaningful interpretations in many of the applications and physical phenomena modeled by zero forcing. For example, in the application of zero forcing to power network monitoring,[1] there are often significant costs associated with the high-speed communication infrastructure between the phase measurement units (PMUs) and the processing stations which collect and manage PMU data; there may also be costs associated with dispatching a technician to regulate or maintain the PMUs and related equipment. In a scenario where these costs outweigh the production costs of the PMUs, an electric power

* Corresponding author.
E-mail addresses: boris.brimkov@rice.edu (B. Brimkov), ivhicks@rice.edu (I.V. Hicks).

[1] See [9] for a more thorough introduction to the power domination problem and its relation to zero forcing.

http://dx.doi.org/10.1016/j.dam.2017.05.016
0166-218X/

company may seek to place all PMUs in a compact, connected region in the network in order to decrease the costs incurred by processing stations, communication infrastructure, and technician travel times. The connected zero forcing number is an upper bound on the number of PMUs required to monitor the network in this scenario, and can be used as a technical tool in the derivation of certain results about connected power domination. As another example, it is often the case that ideas or diseases originate from a single connected source in a social network or geographic region; thus, connected zero forcing may be better suited than zero forcing to model propagation in those scenarios.

As one of the main results of this paper, we establish the NP-completeness of connected zero forcing. Thus, as with zero forcing, this problem cannot be solved efficiently in general, but one can obtain structural results, bounds, and characterizations for specific graphs. To this end, we present several structural results and technical lemmas about connected zero forcing, such as the effects of certain vertex and edge operations on the connected zero forcing number, and properties of vertices which are contained in every connected zero forcing set. We also give two lower bounds on the connected zero forcing number in terms of certain separating sets and induced subgraphs in the graph, and we characterize the connected zero forcing numbers of trees, unicyclic graphs, and some variants of cactus and block graphs. Other theoretical and computational aspects of connected zero forcing have been explored in [12–14,21,40]; some of these results are generalized in the present paper.

Numerous other variants of zero forcing have also been studied, including positive semidefinite zero forcing [5,26,54], Z_q-forcing [17], skew zero forcing [3], signed zero forcing [33], fractional zero forcing [37], and k-forcing [4,19,42]. These variants are typically obtained by modifying the color change rule or by adding certain restrictions to the structure of a zero forcing set; they are often designed to bound other linear algebraic parameters, or introduced as natural graph theoretic extensions of zero forcing. The connected variants of other graph problems, including connected vertex cover [18,41], connected domination [20,23,30,50] and connected power domination [29], have also been extensively studied. Imposing connectivity often fundamentally changes the nature of a problem, including its complexity, structural properties, and applications. For example, while both domination and connected domination are NP-complete [31], the latter is generally harder to solve exactly. This disparity has been attributed to the non-locality of the connected domination problem, since exact algorithms are often unable to capture global properties like connectivity [30]. In some contrast, computational experiments in [13] have shown that algorithms for connected zero forcing are faster, and able to handle larger graphs, than the state-of-the-art zero forcing algorithms. The connected zero forcing number can also be efficiently computed in graphs with polynomially many connected induced subgraphs; this graph class includes arbitrary subdivisions of fixed graphs and graphs with bounded maximum leaf numbers [43].

The remainder of this paper is organized as follows. In the next section, we recall some graph theoretic notions, specifically those related to zero forcing. In Section 3, we present novel structural results about connected zero forcing, as well as some technical lemmas which are used in the sequel. In Section 4, we prove that connected zero forcing is NP-complete. In Section 5, we give closed formulas for the connected zero forcing numbers of several families of tree-like graphs. We conclude with some final remarks and open questions in Section 6.

2. Preliminaries

2.1. Graph theoretic notions

A graph $G = (V, E)$ consists of a vertex set V and an edge set E of two-element subsets of V. The *order* and *size* of G are denoted by $n = |V|$ and $m = |E|$, respectively. Two vertices $v, w \in V$ are *adjacent*, or *neighbors*, if $\{v, w\} \in E$. If v is adjacent to w, we write $v \sim w$; otherwise, we write $v \not\sim w$. The *neighborhood* of $v \in V$ is the set of all vertices which are adjacent to v, denoted $N(v; G)$; the *degree* of $v \in V$ is defined as $d(v; G) = |N(v; G)|$. The minimum degree and maximum degree of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. The dependence of these parameters on G can be omitted when it is clear from the context. Given $S \subset V$, the *induced subgraph* $G[S]$ is the subgraph of G whose vertex set is S and whose edge set consists of all edges of G which have both endpoints in S. The number of connected components of G will be denoted by $c(G)$; an isomorphism between graphs G_1 and G_2 will be denoted by $G_1 \simeq G_2$.

A *leaf*, or *pendant*, is a vertex with degree 1. A *cut vertex* is a vertex which, when removed, increases the number of connected components in G. Similarly, a *cut edge* is an edge which, when removed, increases the number of components of G. A *biconnected component*, or *block*, of G is a maximal subgraph of G which has no cut vertices. An *outer block* is a block with at most one cut vertex. A *unicyclic graph* is a graph with exactly one cycle. A *cactus graph* is a graph in which every block is a cycle or a cut edge, and a *block graph* is a graph in which every block is a clique. For other graph theoretic terminology and definitions, we refer the reader to [11].

2.2. Zero forcing

Given a graph $G = (V, E)$ and a set $S \subset V$ of initially colored vertices, the *color change rule* dictates that at each integer-valued time step, a colored vertex u with a single uncolored neighbor v *forces* that neighbor to become colored; such a *force* is denoted $u \to v$. The *derived set* of S is the set of colored vertices obtained after the color change rule is applied until no new vertex can be forced; it can be shown that the derived set of S is uniquely determined by S (see [2]). A *zero forcing set* is a set whose derived set is all of V; the *zero forcing number* of G, denoted $Z(G)$, is the minimum cardinality of a zero forcing set.

Please cite this article in press as: B. Brimkov, I.V. Hicks, Complexity and computation of connected zero forcing, Discrete Applied Mathematics (2017), http://dx.doi.org/10.1016/j.dam.2017.05.016.